Existence of Equilibrium in Large Matching Markets with Complementarities*

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Abstract

In two-sided matching markets with contracts, the existence of stable outcomes can be guaranteed only under certain restrictions on preferences; the typical restriction is that all agents' preferences are substitutable. We show that, in markets with a continuum of each type of agent, it is only necessary that agents on one side of the market have substitutable preferences to guarantee the existence of a stable outcome. We also consider more general settings with multilateral contracts and no structure on the set of agents, and show that the core is nonempty when there exists a continuum of agents of each type, regardless of agents' preferences. Finally, we show that in settings with bilateral contracts and transferable utility (but an arbitrary contractual network), the existence of competitive equilibria is guaranteed regardless of agents' preferences. We also consider large finite markets, showing that each of the three results above holds approximately in the analogous large finite market.

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1 Introduction

Many markets involve differentiated goods and services, where parties care about characteristics of their business partners. Examples span labor markets, such as hiring associates at a law firm, HMO/hospital networks in health care, supply chain networks in manufacturing, and auctions for telecommunications spectrum. In economics, matching markets are the topic of a large literature, from the Becker (1973) marriage model to more recent contributions that incorporate settings where agents' preferences are heterogeneous,[1] such as the Kelso and Crawford (1982) labor market model and the Hatfield and Milgrom (2005) model of matching with contracts.

However, standard models of matching markets are unable to incorporate a key feature of many settings: *complementarities*. Yet complementarities are an important element of many markets. For example, a startup needs entrepreneurs, programmers, and graphic designers. Similarly, in the design of labor market clearinghouses, couples find certain pairs of positions to be complementary (e.g., positions in the same geographic area). Complementarities are also an important element in matching models with one-dimensional heterogeneity, such as the Kremer (1993) O-ring theory. However, almost all models that allow for heterogeneous preferences assume away most forms of complementarity to ensure the existence of an equilibrium.[2] In fact, it is well-known that equilibria often fail to exist in markets with complementarities.[3] This leaves open the question of how to incorporate complementarities into models with heterogeneous preferences.

We ask whether in markets with a continuum of agents existence of an equilibrium can be guaranteed even with complementarities and heterogeneous preferences. The answer is a qualified yes. We show that equilibria do not always exist in large markets, but do

[1]We use the term heterogeneous preferereneces to mean that preferences may differ across agents. In particular, this includes models with rich heterogenous preferences such as those of Gale and Shapley (1962) and Kelso and Crawford (1982).

[2]See, for example, the work of Kelso and Crawford (1982), Roth (1984b), Gul and Stacchetti (1999, 2000), Hatfield and Milgrom (2005), Sun and Yang (2006, 2009), and Hatfield et al. (2013).

[3]For instance, see the work of Gul and Stacchetti (1999), Hatfield and Kojima (2008), Hatfield and Kominers (2011), and Hatfield et al. (2013).

exist for a much more general class of preferences than in models with discrete agents. In particular, we demonstrate the existence of stable outcomes in applied settings such as matching with couples (Roth and Peranson, 1999; Kojima et al., 2013; Ashlagi et al., 2014), and generalizations of commonly used empirical models of matching (Choo and Siow, 2006; Fox, 2010; Salanié and Galichon, 2011).

Our results can be summarized as follows: We consider models of matching markets akin to Hatfield and Milgrom (2005) and Ostrovsky (2008), but with a continuum of agents. Our first result is that complementarities might preclude the existence of stable outcomes even in large markets. That is, we give a natural example of a market with a continuum of agents and no stable outcome. We then show that in many settings we can guarantee the existence of an equilibrium with a continuum of agents: Our second result demonstrates existence of a stable outcome in two-sided markets when agents on one side all have substitutable preferences. Our third result demonstrates that, in a quasilinear setting, a competitive equilibrium exists with arbitrary preferences. Our fourth result shows that, even with general preferences, and without recourse to a numeraire, the core of a matching market is always non-empty. Finally, we show that our results for continuum models imply approximate existence results for large, finite markets.

This paper is organized as follows. In Section 2 we illustrate our main results with examples, and in Section 3 we describe the relationship of our work to the rest of the literature. In Section 4, we demonstrate the existence of stable outcomes in bilateral matching economies. We then show that core outcomes exist for continuum economies with multilateral contracting in Section 5. Finally, we consider continuum economies with discrete contracting but transferable utility in Section 6. Section 7 concludes.

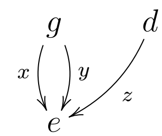

Figure 1: A simple economy. An arrow denotes a contract.

2 Understanding our Results

To illustrate our results, we begin with an example: entrepreneurs and programmers matching to form technology startups. There are three types of agents: entrepreneurs (e), generalist programmers (g), and database programmers (d), with a continuum of equal mass of each type. Entrepreneurs have an idea for a business, but need both types of programming services. For now, transfers between the agents are limited: There are only three contracts available, each of which specifies some standardized compensation for fulfillment of a programming task. There is a continuum of mass 1 of agents of each type. Entrepreneurs may contract for general programming (x) and database programming (y) from the general programmers, or for database programming from the database experts (z), as depicted in Figure 1. Entrepreneurs need both types of programming to create a viable startup, and prefer that the database programming be performed by a specialist. Their preferences over bundles of contracts are $\{x, z\} \succ \{x, y\} \succ \varnothing$. Database programmers would rather contract than not, i.e., $\{z\} \succ \varnothing$, and generalist programmers are only willing to contract if they can sell both of their services, i.e., $\{x, y\} \succ \varnothing$.

The first surprising observation about this example, and our first result, is that there is no stable outcome, *even with a continuum of agents*. Stability is a natural solution concept in this setting: An outcome is stable if it is individually rational (i.e., no agent wishes to unilaterally withdraw from some contracts he currently signs) and there is no blocking set of contracts (i.e., a set of contracts each agent would choose given his current outcome, possibly dropping some contracts he is currently a party to). In this example, in any stable outcome every employed generalist programmer must be signing the only individually rational contract

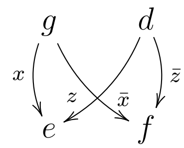

Figure 2: An economy with a stable outcome when there is a continuum of types. An arrow denotes a contract.

bundle $\{x, y\}$. However, if any entrepreneur is engaging in the bundle of contracts $\{x, y\}$, then that entrepreneur would rather drop the y contract and obtain specialized database services instead, moving to their preferred bundle $\{x, z\}$. Such an outcome is not stable, as the generalist programmers are not interested in only selling x, and would rather not transact; but if no agents were transacting, then $\{x, y\}$ would be a blocking set, as both the entrepreneur and the generalist prefer this bundle to nothing. Therefore, even with a continuum of agents, a stable outcome does not exist: simply assuming a large market is not sufficient to guarantee existence of stable outcomes without additional assumptions on preferences. However, the existence of stable outcomes can be shown under substantially more general conditions than in markets with a finite number of agents.

Our second result is that stable outcomes do exist in any large two-sided market where one side has substitutable preferences. This assumption is considerably more general than the standard assumption in the discrete matching literature, namely that all agents have substitutable preferences (Alkan and Gale, 2003; Fleiner, 2003; Hatfield and Milgrom, 2005; Hatfield and Kominers, 2011). In particular, this result implies that a stable matching exists for any large two-sided many-to-one matching market, since unit-demand preferences are a special case of substitutable preferences.

Consider a variation of our previous example: There are still two programmer types, g and d, but now there are two firms, our entrepreneur e and another firm f, who desires at most one contract—either $\bar{x}$ with the generalist programmer, or $\bar{z}$ with the database programmer. We assume that the preferences of f are given by $\{\bar{x}\} \succ \{\bar{z}\} \succ \varnothing$. The generalist programmer now only desires one contract, either x or $\bar{x}$, and has preferences given by $\{x\} \succ \{\bar{x}\} \succ \varnothing$.

The database programmer prefers $\bar{z}$ to z, i.e., has preferences given by $\{\bar{z}\} \succ \{z\} \succ \varnothing$. Finally, the preferences of the entrepreneur e are given by $\{x, z\} \succ \varnothing$. Figure 2 depicts the structure of this example.

In a discrete model with one agent of each type, there is no stable outcome: If both programmers are matched to e, then f can block the match by poaching the database programmer, i.e., $\{\bar{z}\}$ is a blocking set. But $\{\bar{z}, x\}$ is not stable, as it is not individually rational for e. The outcome $\{\bar{z}\}$ is not stable, as f will then wish to lay off the database programmer and hire the generalist programmer, i.e., $\{\bar{x}\}$ is a blocking set. Finally, the outcome $\{\bar{x}\}$ is not stable, as then both programmers would be willing to work for the entrepreneur, i.e., $\{x, z\}$ is a blocking set.

By contrast, if there is a continuum of agents of each type, a stable outcome exists. Suppose there are an equal mass of each type of agent. Half of the programmers of each type work for each type of entrepreneur. In that case, firms of type f are at capacity, and so do not wish to acquire any more programmers from other firms. Likewise, an entrepreneur of type e will not block the match as he is unable to attract a database programmer.

Our third result shows that, if agents have quasilinear preferences over a numeraire commodity, a competitive equilibrium always exists. Thus, in both of the examples given above, if entrepreneurs and programmers have quasilinear preferences over a numeraire, then prices can adjust so that the market clears. In fact, when agents have quasilinear preferences over a numeraire commodity, competitive equilibria exist in general trading networks, not just two-sided (i.e., buyer–seller) economies. Hence, when a continuum of each type of agent is present, no conditions on preferences are required to guarantee the existence of equilibrium in models with a numeraire. By contrast, in models with a discrete number of agents significant restrictions must be placed on preferences in order to ensure the existence of a competitive equilibrium; these restrictions are necessary in both models of exchange economies (Gul and Stacchetti, 1999, 2000; Sun and Yang, 2006, 2009; Baldwin and Klemperer, 2015) and matching models with discrete contractual relationships

(Kelso and Crawford, 1982; Hatfield et al., 2013).[4] Our work also generalizes the existence result of Azevedo et al. (2013), who demonstrate existence in a general equilibrium setting with indivisible goods, but without the rich set of contracts we consider. Moreover, our result implies the existence of stable matchings for the roommate problem (with transfers), first shown by Chiappori et al. (2014).

Finally, our fourth result shows that, even in settings where stable outcomes do not exist, the core of a matching market is always non-empty. Returning to the first startup example, consider the outcome where all entrepreneurs sign the set of contracts $\{x, y\}$ with generalist programmers. Although this allocation is not stable, it is in the core. There is no coalition of agents that can do better, as for an entrepreneur to move to the z contract with the specialized database programmer would require withdrawing from his entire relationship with the generalist programmer. The core is always non-empty in a class of models allowing multilateral contracting, general trading networks, and limitations on transfers.

Our results for continuum economies raise the question of whether approximately stable outcomes exist in large, but finite, matching markets, for an appropriately defined concept of approximate stability. Consider again the example depicted in Figure 2; recall that in the stable outcome in the continuum economy half of the programmers of each type join the entrepreneur and the other half join the firm. Now suppose there is a large, but finite, number k of agents of each type. If k is even, then a stable outcome analagous to the continuum stable outcome exists: $\frac{k}{2}$ of the programmers of each type join the entrepreneur and the other programmers join the firm. If k is odd, no stable outcome exists, but an approximately stable outcome exists, in the sense that there exists an individually rational outcome for which every blocking set contains one particular agent.

Proposition 1 generalizes the example above, showing that, in replicas of a finite economy with substitutable preferences on one side, there is an individually rational outcome that is stable except for a bounded number of agents. Propositions 2 and 5 demonstrate similar

[4]However, in the special case of an auction, i.e., a buyer–seller market with only one seller, a competitive equilibrium always exists: see Bikhchandani and Ostroy (2002).

results for core and competitive equilibrium outcomes in large finite markets. Thus, a mechanism designer can implement allocations where, as in the example above, only a small fraction of agents receive allocations that can lead to blocking pairs. In the case of competitive equilibrium, this is an allocation where, given equilibrium prices, only a small number of agents are assigned bundles that are individually rational but not optimal.

We caution readers that whether approximately stable matchings are a useful solution concept depends on details of the institutional setting. First, if it is easy for agents to find and implement blocking sets, approximately stable outcomes may not be predictive of final market outcomes. Second, from a mechanism design perspective, legal or other constraints may preclude an approximately stable outcome from the set of feasible market outcomes. Finally, computation of approximately stable outcomes must be practical.[5]

Nevertheless, in many settings, approximate stability can be seen as either an intuitively plausible prediction of market outcomes or as a mechanism design goal. As a prediction of market outcomes, approximate stability seems a reasonable prediction when even only mild recontracting costs are present. As a mechanism design goal, approximate stability may be sufficient to satisfy the market designer's goals with respect to fairness or rendering contracting opportunities outside the match impractical for market participants. For these reasons, we see our results on large finite economies as complementary to the literature on the existence of exact stable matchings by offering a practical alternative in applied settings where stable matchings need not exist (e.g., in settings where complementarities are a first-order concern).

[5]Our existence proofs rely on topologicial fixed-point theorems. This may be a challenge for practical implementation because calculating such a fixed point takes exponential time in the worst case. However, practical computational techniques for approximating such fixed points are well-known: see, e.g., Scarf (1967) and Kuhn and MacKinnon (1975), as well as Allgower and Georg (2012) for an overview. Moreover, there are many successful market design applications that rely on heuristics or algorithms with bad worst-case performance, such as the way the NRMP calculates stable matchings with couples (Roth and Peranson, 1999), and fixed-point algorithms used in course allocation (Budish, 2011).

3 Relationship to the Literature and Applications

A number of recent papers that have applied large market ideas to matching (see, e.g., the work of Immorlica and Mahdian (2005), Kojima and Pathak (2009), Lee (2011), Kojima et al. (2013), Ashlagi et al. (2014), Lee and Yariv (2014), and Ashlagi et al. (2015)). Some papers explicitly consider a model with a continuum of agents. Bodoh-Creed (2013), Echenique et al. (2013), and Menzel (2015), like us, consider a model with a continuum of agents on both sides, while Azevedo and Leshno (2011) and Abdulkadiroğlu et al. (2015) have a finite number of firms being matched to a continuum mass of workers. The key difference between our work and these papers is that we focus on the existence of equilibrium in a very general setting. By contrast, other works consider settings without complementarities, where existence was well-known in the discrete case. The focus of Azevedo and Leshno (2011), Bodoh-Creed (2013), Abdulkadiroğlu et al. (2015), and Menzel (2015) is instead in building tractable models in those settings, and applying them to specific problems. Echenique et al. (2013) investigate testable implications of stability.

Concurrent with our work here, Che et al. (2014) considered a model of many-to-one matching with a finite number of firms but a continuum of workers. Each firm demands a measure of workers while each worker has unit demand, i.e., wishes to have at most one job. Firms have very general preferences over the set of workers they hire, allowing for complementarities; nevertheless, they find that a stable outcome always exists. The key substantial difference to our work in Section 4 is that they consider the limit where each firm hires many workers, whereas we consider the limit where there are many workers and many firms. Because they consider a different limit, their conditions for existence are different. For example, their existence result depends on firms having upper hemi-continuous and convex-valued choice correspondences. By contrast, in our limit stable outcomes exist in any many-to-one matching model. The reason is that our limit, with a large number of agents, convexifies aggregate choice correspondences. Nevertheless, their paper is not a particular case of ours, because they consider an infinite number of types, incorporating rich preference structures that are not allowed in our setting.

Nguyen and Vohra (2014) consider existence of approximately stable outcomes. Their main application is resident-hospital matching with couples. They show that it is possible to change the capacities of hospitals so that a stable outcome always exists. They use Scarf's Lemma and combinatorial optimization techniques to tightly bound how much it is necessary to change capacities: For example, in the matching with couples context the total hospital capacity in the market has to increase by at most nine. Their results can be generalized far beyond the couples example. The most substantive assumptions are that all agents are acceptable matches, and that hospital preferences satisfy a generalized responsiveness condition. Our work differs in that we derive existence of stable outcome results in very general settings, while they give tight bounds on the capacity increases necessary to restore stability.

Our paper relates to two applied threads of the matching literature. One particular case of our framework is two-sided matching with couples. This is an important applied market design problem, as labor market clearinghouses, such as the National Resident Matching Program (NRMP), take couples' preferences into account. Indeed, matching couples was one of the central issues of the redesign of this clearinghouse, described in Roth and Peranson (1999). Although in the NRMP data stable matchings typically exist, it is well-known that existence of a stable matching with couples cannot be guaranteed in finite markets (Klaus and Klijn, 2005). Recently, Kojima et al. (2013), and Ashlagi et al. (2014) have shown that in a large class of large, finite markets, stable matchings exist with high probability. However, both of these works assume that the fraction of participants that are members of couples in the population converges to zero as the market grows. Moreover, Ashlagi et al. (2014) give a strong negative result, providing an example of large discrete markets with a fixed proportion of couples that have no stable matching. Our results show that stable matchings with couples always exist in a model with a continuum of agents. Moreover, in markets with a large finite number of agents, we prove that an approximately stable matching always exists: while we cannot explain the fact that stable matchings often exist in the data, our results guarantee that non-existence of a stable matching does not pose a problem to clearinghouses such as the NRMP, as an approximately stable matching always exists. Thus, our results are complementary to those of Kojima et al. (2013) and Ashlagi et al. (2014), and together make a strong case for the robustness of the NRMP market design.

An important implication of our result is that empirical models of matching with transfers can be extended to large market settings where preferences are not necessarily substitutable (Choo and Siow, 2006; Fox, 2010). Our result shows that equilibria exist in natural extensions of these empirical models, allowing the incorporation of significant features of real markets such as complementarities between workers or firms supplying production inputs.

Finally, from a technical perspective, this is one of the first papers to apply techniques from general equilibrium to matching, such as topological fixed point theorems and approximate equilibrium concepts. These techniques have been important in showing the existence of equilibrium (Arrow and Debreu, 1954), including in settings with non-convex preferences (Aumann, 1964; Starr, 1969) and complementary demand for indivisible goods (Azevedo et al., 2013). By contrast, the standard approach to prove existence in the matching literature is to use order-theoretic fixed-point theorems, such as Tarsi's fixed point theorem. We hope that, as in general equilibrium, this approach will be useful in applied matching problems with complementary preferences.

4 Stable Outcomes in Large Economies

4.1 Framework

There is a finite set B of *buyer types* and a finite set S of *seller types*; for each agent type $i \in I \equiv B \cup S$, there exists a mass θ^i of agents of type i. There also exists a finite set X of *contracts*, and each contract $x \in X$ is associated with a buyer type $b \in B$, denoted $\mathsf{b}(x)$, and a seller type $s \in S$, denoted $\mathsf{s}(x)$.

For a set of contracts $Y \subseteq X$, we let $\mathsf{b}(Y) \equiv \cup_{y \in Y}\{\mathsf{b}(y)\}$ and $\mathsf{s}(Y) \equiv \cup_{y \in Y}\{\mathsf{s}(y)\}$. We also let $Y_i \equiv \{y \in Y : i \in \mathsf{b}(Y) \cup \mathsf{s}(Y)\}$ denote the set of contracts in Y associated with agents of type i.

4.1.1 Preferences

Each type of agent $i \in I$ has strict preferences $\succ_i$ over sets of contracts involving that agent. We naturally extend preference relations to subsets of X: for $Y, Z \subseteq X$, we write $Y \succ_i Z$ if and only if $Y_i \succ_i Z_i$.

For any agent type $i \in I$, the preference relation $\succ_i$ induces a *choice function*

$$C^i(Y) \equiv \max_{\succ_i}\{Z \subseteq Y : x \in Z \Rightarrow i \in \{\mathsf{s}(x), \mathsf{b}(x)\}\}$$

for any $Y \subseteq X$.[6]

The notion of substitutability has been key in assuring the existence of stable outcomes in settings with a finite number of agents.[7] An agent type $i \in I$ has substitutable preferences if, when presented with a larger choice set, any previously rejected contract is still rejected.

Definition 1. An agent type $i \in I$ has *substitutable* preferences if for all $x, z \in X$ and $Y \subseteq X$, if $z \notin C^i(Y \cup \{z\})$, then $z \notin C^i(\{x\} \cup Y \cup \{z\})$.

[6] Here, we use the notation $\max_{\succ_i}$ to indicate that the maximization is taken with respect to the preferences of agent i.

[7] In the setting of many-to-many matching with contracts, substitutable preferences are both sufficient (Roth, 1984b; Echenique and Oviedo, 2006; Klaus and Walzl, 2009; Hatfield and Kominers, 2011) and necessary (Hatfield and Kominers, 2011) to guarantee the existence of stable outcomes. In the setting of many-to-one matching with contracts, substitutability of preferences is sufficient (Hatfield and Milgrom, 2005), but not necessary (Hatfield and Kojima, 2008, 2010); however, if each contract specifies a unique buyer-seller pair, preference substitutability is necessary (Hatfield and Kojima, 2008). Similarly, in settings with transferable utility, substitutability is both sufficient to guarantee the existence of competitive equilibria (Kelso and Crawford, 1982; Gul and Stacchetti, 1999; Sun and Yang, 2006; Hatfield et al., 2013) and necessary (Gul and Stacchetti, 1999; Hatfield and Kojima, 2008; Hatfield et al., 2013).

4.1.2 Outcomes

We let m^i_Z denote the mass of agents of type $i \in I$ who engage in contracts $Z \subseteq X_i$; thus, $m^i \in [0, \theta^i]^{\mathcal{P}(X_i)}$, where $\mathcal{P}(X_i)$ is the power set of X_i. The *supply* of a contract $x \in X$ is given by

$$m_x^{\mathsf{s}(x)} \equiv \sum_{\{x\} \subseteq Z \subseteq X_{\mathsf{s}(x)}} m_Z^{\mathsf{s}(x)}$$

while *demand* is given by

$$m_x^{\mathsf{b}(x)} \equiv \sum_{\{x\} \subseteq Z \subseteq X_{\mathsf{b}(x)}} m_Z^{\mathsf{b}(x)}.$$

We may now define an outcome for this economy as a vector of contract allocations for each type of agent such that supply equals demand.

Definition 2. An *outcome* is a vector $((m^b)_{b \in B}, (m^s)_{s \in S})$, where $m^i \in [0, \theta^i]^{\mathcal{P}(X_i)}$ for each $i \in I$, such that

1. For all $i \in I$, $\sum_{Z \subseteq X_i} m^i_Z = \theta^i$, and

2. For all $x \in X$, $m_x^{\mathsf{s}(x)} = m_x^{\mathsf{b}(x)}$.

The first condition of Definition 2 ensures that the total mass of type i agents participating in some subset of contracts is equal to the total mass of those type of agents in the economy; note that an agent does not participate in any contract if he participates in the empty set of contracts. The second condition ensures that for each contract x, the mass of sellers participating in x is the same as the mass of buyers participating in x.

4.2 Existence

As is standard in matching theory, we define an equilibrium as a *stable* outcome.

Definition 3. An outcome m is *stable* if it is

1. *Individually rational*, i.e., for all $i \in I$ and $Z \subseteq X_i$, if $Z \neq C^i(Z)$, then $m^i_Z = 0$, and

2. *Unblocked*, i.e., there does not exist a nonempty multiset[8] Z composed of the elements of X such that

   (a) There exists a partition[9] of Z into sets $\{Z^{\mathfrak{b}}\}_{\mathfrak{b}\in\mathfrak{B}}$ such that for each $Z^{\mathfrak{b}}$ there exists a buyer type $b \in B$ such that $\{b\} = \mathsf{b}(Z^{\mathfrak{b}})$ and associated set $Y^{\mathfrak{b}} \subseteq X_b \smallsetminus Z^{\mathfrak{b}}$ such that

      i. $m^b_{Y^{\mathfrak{b}}} > 0$, and

      ii. $Z^{\mathfrak{b}} \subseteq C^b(Z^{\mathfrak{b}} \cup Y^{\mathfrak{b}})$.

   (b) There exists a partition of Z into sets $\{Z^{\mathfrak{s}}\}_{\mathfrak{s}\in\mathfrak{S}}$ such that for each $Z^{\mathfrak{s}}$ there exists a seller type $s \in S$ such that $\{s\} = \mathsf{s}(Z^{\mathfrak{s}})$ and associated set $Y^{\mathfrak{s}} \subseteq X_s \smallsetminus Z^{\mathfrak{s}}$ such that

      i. $m^s_{Y^{\mathfrak{s}}} > 0$, and

      ii. $Z^{\mathfrak{s}} \subseteq C^s(Z^{\mathfrak{s}} \cup Y^{\mathfrak{s}})$.

This definition of stability is equivalent to the standard definition from the matching literature (see, e.g., Hatfield and Milgrom (2005)). Individual rationality requires that for any set $Z \subseteq X_i$ of contracts that a positive mass of agents of type $i \in I$ engage in, no proper subset $\tilde{Z} \subsetneq Z$ is preferred to Z. Unblockedness requires that no block exists, but a block is now comprised of a multiset, instead of a set, as a block may now require multiple agents of the same type to choose different parts of the blocking multiset Z.[10] Hence, the multiset Z is a block so long as both

1. It can be decomposed into sets $\{Z^{\mathfrak{b}}\}_{\mathfrak{b}\in\mathfrak{B}}$ such that, for each set $Z^{\mathfrak{b}}$, there exists a positive measure of the associated buyer type who would choose that set (given the current outcome), and

2. It can also be decomposed into (in general, different) sets $\{Z^{\mathfrak{s}}\}_{\mathfrak{s}\in\mathfrak{S}}$ such that, for each

[8]The notion of a multiset generalizes the notion of a set in that it allows for elements to appear more than once.

[9]A *partition* of a multiset Z is a multiset $\{Z^{\mathfrak{i}}\}_{\mathfrak{i}\in\mathfrak{I}}$ such that $\biguplus_{\mathfrak{i}\in\mathfrak{I}} Z^{\mathfrak{i}} = Z$, where $\uplus$ is the standard multiset sum.

[10]Note that, in our setting, contracts do not uniquely identify an agent, but rather an agent type.

$$
\begin{tikzcd}
s \arrow[d, bend right, "x"'] \arrow[d, bend left, "y"] \\
b
\end{tikzcd}
$$

Figure 3: A simple economy. An arrow denotes a contract.

set Z^{s}, there exists a positive measure of the associated seller type that would choose that set (given the current outcome).

We illustrate the model with a simple example.

Example 1. Consider a simple economy where $B = \{b\}$, $S = \{s\}$ and $\theta^b = \theta^s = 1$. Let $X = \{x, y\}$ where $\mathsf{b}(x) = \mathsf{b}(y) = b$ and $\mathsf{s}(x) = \mathsf{s}(y) = s$, illustrated in Figure 3. Let preferences be given by

$$
\begin{aligned}
b &: \{x, y\} \succ \varnothing, \\
s &: \{x\} \succ \{y\} \succ \varnothing.
\end{aligned}
$$

The only stable outcome is given by $m^b_{\{x,y\}} = \frac{1}{2}, m^b_{\varnothing} = \frac{1}{2}, m^s_{\{x\}} = \frac{1}{2}, m^s_{\{y\}} = \frac{1}{2}$, with all other entries of the matrix m being zero. Note that to show the outcome $m = 0$ is not stable in our setting requires the full generality of Definition 3, where we let $Z = \{x, y\}$ and consider the partition $\{\{x, y\}\}$ for buyers and the partition $\{\{x\}, \{y\}\}$ for sellers.

We now state the main theorem of this section.

Theorem 1. *If buyers' preferences are substitutable, then a stable outcome exists.*

To prove Theorem 1, we construct a generalized Gale-Shapley operator. Let $O^B \in [0, \infty)^X$ denote an *offer vector* for the buyers, i.e., the mass of each contract the buyers have access to.

Suppose that b has preferences given by

$$
Y^K \succ_b \ldots \succ_b Y^k \succ_b \ldots \succ_b Y^1 \succ_b \varnothing
$$

over all individually rational subsets of X_b. We define $h^b_{Y^k}(O^B)$ inductively, $k = K, \ldots, 0$ as

$$h^b_{Y^k}(O^B) \equiv \min\left\{\theta^b - \sum_{\tilde{k}>k}^{K} h^b_{Y^{\tilde{k}}}(O^B), \min_{x \in Y^k}\left\{O^B_x - \sum_{\tilde{k}>k}^{K} h^b_{Y^{\tilde{k}}}(O^B)\mathbb{1}\{x \in Y^{\tilde{k}}\}\right\}\right\}. \qquad (1)$$

The first term of the minimand is the remaining mass of agents of type b who are not yet assigned via the inductive process. The second term of the minimand is the amount of the set Y^k still available from the offer vector O^B given the mass of each contract in Y^k taken at an earlier step of the inductive process. Intuitively, each buyer type is assigned the maximal amount of that buyer type's favorite set of contracts Y^K as possible; having done so, that buyer type is then assigned the maximal amount of that buyer type's second favorite set of contracts Y^{K-1} from what is left, and so on. We may then define the choice function for a type of buyer b as

$$\bar{C}^b_x(O^B) \equiv \sum_{\{x\} \subseteq Y \subseteq X_b} h^b_Y(O^B)$$

given an offer vector O^B, i.e., $\bar{C}^b_x(O^B)$ is the mass of x contracts chosen by buyers of type b when these buyers have access to O^B.[11] We define $h^s(O^S)$ for each seller and $\bar{C}^s_x(O^S)$ for each seller analogously. This formulation is equivalent to the usual formulation in finite economies, where O^B is an offer set and the choice function of the buyers is just the union of the choice function of each buyer.

[11] We use the $\bar{C}^b$ notation, as opposed to C^b, to denote that the choice is with respect to all buyers of type b, not just one buyer of type b.

We can now define the following generalized Gale-Shapley operator[12]

$$\Phi(O^B, O^S) \equiv (\Phi^B(O^S), \Phi^S(O^B)) \tag{2}$$
$$\Phi^B_x(O^S) \equiv \bar{C}^{\mathsf{s}(x)}_x((O^S_{X \smallsetminus \{x\}}, \theta^{\mathsf{s}(x)}))$$
$$\Phi^S_x(O^B) \equiv \bar{C}^{\mathsf{b}(x)}_x((O^B_{X \smallsetminus \{x\}}, \theta^{\mathsf{b}(x)})).$$

At each step of the operator, the mass of contract x available to the buyers, O^B_x, is defined by the mass of contract x that sellers would be willing to take if $\theta^{\mathsf{s}(x)}$ of the contract x (i.e., the maximum amount sellers could demand) was available and a mass of every other contract y equal to O^S_y was available.

Since $\bar{C}^b(\cdot)$ and $\bar{C}^s(\cdot)$ are continuous functions for all $b \in B$ and $s \in S$, it follows immediately that Φ is a continuous function from $(\times_{x\in X}[0, \theta^{\mathsf{b}(x)}]) \times (\times_{x\in X}[0, \theta^{\mathsf{s}(x)}])$ to $(\times_{x\in X}[0, \theta^{\mathsf{b}(x)}]) \times (\times_{x\in X}[0, \theta^{\mathsf{s}(x)}])$. Hence, by Brouwer's fixed point theorem, there exists a fixed point.

To complete the proof, all that is necessary is to ensure that fixed points of Φ do, in fact, correspond to stable outcomes, which is established by the following lemma.

Lemma 1. *Suppose that $(O^B, O^S) = \Phi(O^B, O^S)$. Then if buyers' preferences are substitutable, $((h^b(O^B))_{b\in B}, (h^s(O^S))_{s\in S})$ is a stable outcome.*

[12]Note that this operator is not a direct analogue of the generalized Gale-Shapley operator of Hatfield and Milgrom (2005) and Hatfield and Kominers (2012). The analogue to our operator in the discrete setting is given by

$$\Phi(X^B, X^S) \equiv (\Phi^B(X^S), \Phi^S(X^B))$$
$$\Phi^B(X^S) \equiv \{x \in X : x \in C^S(X^S \cup \{x\})\}$$
$$\Phi^S(X^B) \equiv \{x \in X : x \in C^B(X^B \cup \{x\})\}.$$

When preferences of both buyers and sellers are substitutable, this operator is also monotonic, implying the existence of fixed points by Tarski's theorem. Furthermore, a stronger result regarding the relationship between fixed points and stable outcomes can be shown for this operator than the operator of Hatfield and Milgrom (2005) and Hatfield and Kominers (2011, 2012): In particular, there exists a *one-to-one* correspondence between fixed points and stable outcomes when all agents' preferences are substitutable. Moreover, if (X^B, X^S) is a fixed point, then $X^B \cap X^S$ is a stable outcome, $X^B \smallsetminus X^S$ is the set of contracts desired by the sellers but rejected by the buyers (at the outcome $X^B \cap X^S$), $X^S \smallsetminus X^B$ is the set of contracts desired by the buyers but rejected by the sellers (at the outcome $X^B \cap X^S$), and $X \smallsetminus (X^B \cup X^S)$ is the set of contracts rejected by both buyers and sellers (at the outcome $X^B \cap X^S$).

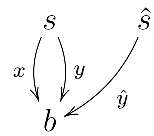

Figure 4: An economy without a stable outcome. An arrow denotes a contract.

Proof. See Appendix A. $\square$

Stable outcomes correspond to fixed points of the generalized Gale-Shapley operator as for any fixed point (O^B, O^S), if Z blocks $((h^b(O^B))_{b\in B}, (h^s(O^S))_{s\in S})$, then for each $z \in Z$, the lowest utility buyers of the associated type $\mathsf{b}(z)$ will choose z from their current set of contracts and z, as the preferences of each buyer type are substitutable. But then each seller must have access to all the contracts in Z; but if Z blocks $((h^b(O^B))_{b\in B}, (h^s(O^S))_{s\in S})$, then some measure of each of the associated seller types will choose all of the corresponding contracts in Z, implying that (O^B, O^S) is not a fixed point.

While substitutability of buyers' preferences is enough to ensure that a stable outcome exists, it is not sufficient for any of the standard structural results on the set of stable outcomes. It is straightforward to construct an example of a many-to-one market where the set of stable outcomes does not form a lattice (in the usual way) and in which the conclusion of the rural hospitals theorem of Roth (1986) does not hold.

However, if the preferences of both sides are not substitutable, then a stable outcome does not necessarily exist, even when there is a continuum of agents and contracts are bilateral.

Example 2. Suppose that $S = \{s, \hat{s}\}$ and $B = \{b\}$ (with $\theta^s = \theta^{\hat{s}} = \theta^b = 1$) and suppose that $X = \{x, y, \hat{y}\}$, where $\mathsf{s}(x) = \mathsf{s}(y) = s$, $\mathsf{s}(\hat{y}) = \hat{s}$, and $\mathsf{b}(x) = \mathsf{b}(y) = \mathsf{b}(\hat{y}) = b$, which is depicted in Figure 4.

Let the preferences of the three agents be given by:

$$s : \{x, y\} \succ_s \varnothing$$

$$\hat{s} : \{\hat{y}\} \succ_{\hat{s}} \varnothing$$

$$b : \{x, \hat{y}\} \succ_b \{x, y\} \succ_b \varnothing$$

No stable outcomes exist. It is immediate that in any stable outcome, individual rationality imposes that $m^s_x = m^s_y$ and that $m^b_y + m^b_{\hat{y}} = m^b_x$. Suppose that $m^s_x = 0$; then $m^b_{\{x,y\}} = m^s_{\{x,y\}} = 0$ and $\{x, y\}$ is a block. Suppose that $m^s_x > 0$; then $m^s_{\{x,y\}} = m^b_{\{x,y\}} > 0$ and $\{\hat{y}\}$ is a block.

The above example shows that stable outcomes do not necessarily exist when preferences of agents on both sides of the market are not substitutable, even when a continuum of agents is present.[13] In Example 2, the key issue is that, when considering blocking (multi)sets, we allow buyer b to break one of his contractual obligations (in this case, dropping y) without affecting the other contracts he has access to; however, since seller s has non-substitutable preferences, when seller s no longer has access to contract y, he also no longer wants to participate in x (which, since b does not have substitutable preferences, would imply that b no longer wishes to agree to y or $\hat{y}$).[14] However, a core outcome, as classically defined, does exist in Example 2.[15] We investigate the existence of the core in large economies in Section 5 below.

Furthermore, even when all agents' preferences are substitutable, the existence of stable outcomes relies on the acyclic nature of the network structure. Consider the setting of Ostrovsky (2008) and Hatfield and Kominers (2012); if we redefine the contract y in Exam-

[13]Example 2 is generic, in the sense that it does not rely on a particular specification of θ. So long as the mass of each type of agent is positive, the model specified in the example above will not have a stable outcome.

[14]However, it is not necessary that a buyer-seller pair have multiple possible contracts between them (as they do in Example 2) in order to construct an example where no stable outcome exists.

[15]The core outcome is given by $m^s_{\{x,y\}} = m^b_{\{x,y\}} = 1$ and $m^i_z = 0$ otherwise. This outcome is in the core as no coalition can improve their joint outcome; b is only better off if he obtains both x and $\hat{y}$ (and drops contract y), but this requires s to agree even though such an outcome is not even individually rational for s.

ple 2 so that $\mathsf{s}(y) = b$ and $\mathsf{b}(y) = s$, then the preferences of each agent are fully substitutable in the sense of Hatfield and Kominers (2012), but the network structure is cyclic. However, since there is no stable outcome in Example 2, simply relabeling the buyer and seller of a particular contract should not induce a given outcome to become stable. Hence, acyclicity of the network structure remains a necessary condition to guarantee the existence of a stable outcome even in the presence of substitutable preferences and a continuum of agents.

4.3 Stable Outcomes in Large Finite Economies

We now extend our model to consider the case where there is a large but finite number of agents. Define a *finite economy* as a vector $n = (n^i)_{i \in I}$, specifying a non-negative integer number of agents of each type. We denote by $|n|$ the number of agents in economy n. For any positive integer k, we will refer to the economy $k \cdot n$ as the *k-replica* of economy n.

A vector $(m^i_Z)_{i \in I, Z \subseteq X_i}$ is a *stable outcome* of the finite economy n if it is a stable outcome in the continuum model with $\theta = n$ and all of the coordinates of m are integers. We say that the finite economy n has an outcome that is *stable excluding α agents* if there exists a finite economy $\bar{n}$ with a stable outcome such that $\bar{n}^i \leq n^i$ for all $i \in I$ and $|n| \leq |\bar{n}| + \alpha$. Intuitively, n has an outcome that is stable excluding α agents if there exists another finite economy $\bar{n}$ created by excluding up to α agents from n such that $\bar{n}$ has a stable outcome. Essentially, if n has an outcome that is stable excluding α agents, then there exists a feasible outcome for n such that every agent receives an individually rational allocation and any blocking set must involve a contract with at least one of those α agents. We have the following result.

Proposition 1. *Consider a finite economy n, and assume that all buyers have substitutable preferences. There exist positive integers α and β such that:*

1. *Any replica of n has an outcome that is stable excluding α agents.*
2. *For any k that is an integer multiple of β, the k-replica of n has a stable outcome.*

Proof. See Appendix A. $\square$

The first part of the proposition shows that, regardless of the size of a replica, it is always possible to exclude a fixed, finite number of participants and achieve a stable outcome. In particular, as the size of the replica grows, the fraction of agents who have to be excluded is of the order of $\frac{1}{k}$. The second part of the proposition shows that, in any replica that is a multiple of some integer, an exact stable outcome exists.

We now consider an example to illustrate three points: approximately stable outcomes are a reasonable equilibrium prediction in a large market, they can be a compelling market design objective, and they can be applied to situations where the existence of an exact stable outcome depends on restrictive assumptions.

Example 3. In this example, we apply our model to the setting of school choice matching with diversity constraints discussed by Abdulkadiroğlu (2005), Hafalir et al. (2013), Kominers and Sönmez (2014), and Fragiadakis and Troyan (2015), among others. There are two types of students: high-ability students, denoted by h, and low-ability ability students, denoted by ℓ. There are two schools: a school subject to diversity constraints, and a school not subject to diversity constraints. We model these schools by assuming there are two types of "school agents", c and $\bar{c}$, corresponding, respectively, to the two schools in this market; hence, as the market grows large the size of each school will increase (by allowing the number of agents of types c and $\bar{c}$ to increase). The set of contracts is the set of student–school agent pairs, i.e., $X = \{(h, c), (\ell, c), (h, \bar{c}), (\ell, \bar{c})\}$.

The constrained school, i.e., the school corresponding to agents of type c, is required to admit at least as many low-ability students as high-ability students, and so we model this requirement by letting the preferences of an agent of type c be given by

$$c : \{(h, c), (\ell, c)\} \succ \{(\ell, c)\} \succ \varnothing.$$

Meanwhile, the unconstrained school, i.e., the school corresponding to agents of type $\bar{c}$, has no restrictions on whom it may admit, and prefers high-ability students to low-ability students.

We model this by letting the preferences of an agent of type $\bar{c}$ be given by

$$c : \{(h, \bar{c})\} \succ \{(\ell, \bar{c})\} \succ \varnothing.$$

The preferences of the high-ability and low-ability students are given by

$$h : \{(h, c)\} \succ \{(h, \bar{c})\} \succ \varnothing$$
$$\ell : \{(\ell, \bar{c})\} \succ \{(\ell, c)\} \succ \varnothing.$$

Consider first a finite economy $(1, 1, 1, 1)$, i.e., an economy where there is one agent of each type.[16] Then no stable outcome exists. If both students are matched to the constrained school (i.e., the outcome $\{(h, c), (\ell, c)\}$), then the unconstrained school can block the outcome by attracting the low-ability student (i.e., $\{(\ell, \bar{c})\}$ is a blocking set). But the outcome $\{(h, c), (\ell, \bar{c})\}\}$ is also not stable, as once the low-ability student attends the small school, the constrained school violates its diversity constraint; hence, the constrained school then has to reject the high-ability student, i.e., $\{(h, c), (\ell, \bar{c})\}\}$ is not individually rational. Nor is the outcome $\{(\ell, \bar{c})\}$ stable, as then the high-ability student wishes to attend $\bar{c}$, i.e., $\{(h, \bar{c})\}$ is a blocking set. Finally, $\{(h, \bar{c})\}$ is not stable, as the large school can attract both students, i.e., $\{\{(h, c), (\ell, c)\}\}$ is a blocking set.[17]

Moreover, no stable outcome exists for any economy of the form $k \cdot (1, 1, 1, 1)$, where k is an odd integer. For example, if $k = 101$, the outcome where the constrained school has 51 students of each type and the unconstrained school has 50 students of each type is unstable, as the unconstrained school will wish to attract one more low-ability student, and such a student will wish to attend the unconstrained school.

Consider now the continuum model, where $\theta = (1, 1, 1, 1)$. Then a stable outcome m exists, where m is given by $m^h_{\{(h,c)\}} = m^h_{\{h,\bar{c}\}} = \frac{1}{2}, m^\ell_{\{(\ell,c)\}} = m^\ell_{\{\ell,\bar{c}\}} = \frac{1}{2}, m^c_{\{(h,c),(\ell,c)\}} = m^c_\varnothing = \frac{1}{2}$

[16]Note that this corresponds to there being two “seats” at the constrained school but only seat at the unconstrained school.

[17]The only other individually rational outcome, $\varnothing$, is also blocked by $\{(h, c), (\ell, c)\}$.

and $m^{\bar{c}}_{\{(h,\bar{c})\}} = m^{\bar{c}}_{\{(\ell,\bar{c})\}} = \frac{1}{2}$, with all other entries of the matrix m being zero. That is, it is a stable outcome for half of the students of each type to attend the school subject to diversity constraints, and for half of the students of each type to attend the unconstrained school. In this outcome, the unconstrained school is at capacity, and so does not wish to poach low-ability students from the constrained school; moreover, all of the high-ability students are either at the unconstrained school or prefer their current placement to the unconstrained school. The constrained school is under capacity, but is up against its diversity constraint; hence, it is only willing to accept low-ability students, but all such students are already either at the constrained school or they prefer their current placement to the constrained school.

Consider now a k-replica of the $(1, 1, 1, 1)$ economy. If k is even, the continuum stable outcome corresponds to an exact stable outcome of the k-replica economy. Hence, we have that $\beta = 2$ in Proposition 1 for this economy. Moreover, we can always find a feasible outcome for the finite economy for which at most one student is unmatched, i.e., $\alpha = 1$ in Proposition 1 for this economy. Hence, when k is odd, it is a plausible prediction that each school will be matched to about $\frac{k}{2}$ students of each type.

Moreover, an approximately stable outcome is also a reasonable allocation from a market mechanism design perspective. A clearinghouse could not ensure finding an exactly stable outcome in this setting; however, the clearinghouse could implement an approximately stable outcome. Which approximately stable outcome to implement depends on institutional details of the application in question: One option is to allow the constrained school to have an extra high-ability student. Another option would be to artificially reduce the capacity of the unconstrained school by one.[18]

This result also sheds light on the existence of stable outcomes in matching markets with couples. Roth (1984a) first noted that couples may provide a challenge to the National Resident Matching Program, as the preferences of couples are not substitutable, and hence

[18]Budish (2011) proposes a similar idea, where he suggests for the problem of course allocation to slightly lower the capacity for some classes in order to implement approximate *competitive equilibrium from equal incomes*.

the existence of a stable outcome in finite markets is not guaranteed.[19,20] Nevertheless, stable outcomes do exist in practice, as noted by Roth (2002). Our work here helps us to understand the prevalence of stable outcomes in such economies, by demonstrating that in large markets, the number of instabilities in the market is likely to be very small (compared to the size of the market).[21] Moreover, this result guarantees that, even if a stable outcome cannot be found, it is always possible to find an approximately stable outcome.

5 The Core in Large Economies

5.1 Framework

In this section, we allow for more complex contractual structures. In particular, there is a finite set I of *agent types*; for each $i \in I$, there exists a mass θ^i of agents of type i. There also exists a finite set of *roles*, $\mathcal{X}$, and each role $\chi \in \mathcal{X}$ is identified with a unique agent type $\mathsf{a}(\chi)$; for a set of roles $\mathcal{Y} \subseteq \mathcal{X}$, we let $\mathsf{a}(\mathcal{Y}) = \cup_{y \in \mathcal{Y}}\{\mathsf{a}(y)\}$ and denote by $\mathcal{Y}_i$ the set of roles associated with agent i, $\{y \in \mathcal{Y} : \mathsf{a}(y) = i\}$. Each agent type $i \in I$ is endowed with a strict preference order $\succ_i$ over roles in $\mathcal{X}_i$. A *contract* x is a set of roles, i.e., $x \subseteq \mathcal{X}$; we denote by X the set of all contracts. Furthermore, each contract is composed of contract-specific roles, i.e., $x \cap y = \varnothing$ for all distinct $x, y \in X$.[22]

For ease of exposition, we assume that for each agent type $i \in I$, there exists an outside option, i.e., a role o^i and a contract $o^i \equiv \{o^i\}$.

Definition 4. An *outcome* is a vector $(m^i)_{i \in I}$, where $m^i \in [0, \theta^i]^{\mathcal{X}_i}$, such that

1. For all $i \in I$, $\sum_{\chi \in \mathcal{X}_i} m^i_\chi = \theta^i$, and

[19] The preferences of a couple are not typically substitutable, as, for instance, the husband may reject a position at a hospital in Los Angeles until his wife receives an offer at another hospital in Los Angeles.

[20] This observation has generated an extensive literature on the types of preferences for couples for which stable matches may be guaranteed to exist: see Klaus et al. (2007), Klaus and Klijn (2007), and Haake and Klaus (2009).

[21] See also work by Kojima et al. (2013) and Ashlagi et al. (2014), who show that the probability of a stable outcome approaches 1 as the market grows large under certain assumptions on how the market grows.

[22] Since contractual arrangements may involve many agents of the same type, this model is strictly more general than one where agents just have preferences over contracts.

2. For all $x \in X$, for all $\mathscr{x}, \mathscr{y} \in x$, $m_{\mathscr{x}}^{\mathsf{a}(\mathscr{x})} = m_{\mathscr{y}}^{\mathsf{a}(\mathscr{y})}$.

The first condition ensures that each type of agent is fully assigned to some role (possibly the outside option). The second condition ensures that "supply meets demand", that is, for each contract, each role has an equal measure of agents (of the appropriate type) performing that role.

We can now define the core in our setting.

Definition 5. An outcome $(m^i)_{i \in I} \in \times_{i \in I} [0, \theta^i]^{\mathcal{X}_i}$ is in the *core* if there does not exist a contract $z \in X$ such that, for each $i \in \mathsf{a}(z)$, there exists $\mathscr{x} \in \mathcal{X}_i$ such that

1. $m_{\mathscr{x}}^i > 0$ and

2. for each $\mathscr{z} \in z \cap \mathcal{X}_i$, $\mathscr{z} \succ_i \mathscr{x}$.

For an outcome to be in the core, it must be that there does not exist an alternative contract z such that, for every role $\mathscr{z}$ associated with z, we can find a positive measure of agents of the appropriate type who are willing to fulfill that role.

Note that we have assumed that each agent demands only one role; one could consider a more complex model in which agents may demand multiple roles. However, such an economy can be reduced to a unit-demand economy where each set of roles in the multiunit demand economy is defined as a single role in the induced unit-demand economy. Moreover, any element of the core of the induced unit-demand economy corresponds to an element of the core of the multiunit demand economy. See Appendix B.2 for a formal construction of this argument.

5.2 Existence

We now state the main result of this section.

Theorem 2. *A core outcome exists.*

Let the preferences of each $i \in I$ be denoted

$$\chi^{N_i} \succ_i \chi^{N_i - 1} \succ_i \ldots \succ_i \chi^1 \succ_i \chi^0$$

where $\chi^0 = o^i$. We can then define a choice function over an offer vector $O^i \in [0, \theta^i]^{X_i}$ inductively, $n = N^i, \ldots, 0$ as

$$\bar{C}^i_{\chi^n}(O^i) \equiv \min \left\{ O^i_{\chi^n}, \theta^i - \sum_{\hat{n}>n}^{N^i} \bar{C}^i_{\chi^{\hat{n}}}(O^i) \right\}, \tag{3}$$

where $\bar{C}^i_y(O^i) \equiv 0$ for all $y \in X_i$ such that $o^i \succ_i y$. The first term of the minimand is the amount of role χ^n available to agents of type i, and the second term of the minimand is the remaining measure of agents of type i who have not obtained a role they prefer. Intuitively, each type of agent obtains as much of his favorite role as possible; having done so, that type of agent then obtains as much of his second favorite role as possible, and so on.

We now define the generalized Gale-Shapley operator as

$$\Xi^i_\chi((O^i)_{i \in I}) \equiv \begin{cases} \theta^i & \text{if } \chi = o^i \\ \min_{y \in x \setminus \{\chi\}} \bar{C}^{\mathsf{a}(y)}_y((O^{\mathsf{a}(y)}_{X_i \setminus \{y\}}, \theta^{\mathsf{a}(y)})) & \text{otherwise, where } \chi \in x. \end{cases} \tag{4}$$

In each iteration of the operator, the measure of role χ available to agents of type $i = \mathsf{a}(\chi)$ is determined by the minimum of the measures of the other roles associated with that contract desired by other types of agents, given the other opportunities available to those types of agents.[23]

Since $\bar{C}^i$ is a continuous function for each $i \in I$, it follows immediately that Ξ is a continuous function from $\times_{i \in I}[0, \theta^i]^{X_i}$ to $\times_{i \in I}[0, \theta^i]^{X_i}$. Hence, by Brouwer's fixed point theorem, there exists a fixed point. To complete the proof, all that is necessary is to ensure that fixed points of Ξ do, in fact, correspond to core outcomes, which is established by the

[23]For the outside option o^i, agent i always has enough of the outside option available that the full mass of agents of type i may choose it.

following lemma.

Lemma 2. *Suppose that $\Xi((O^i)_{i\in I}) = (O^i)_{i\in I}$. Then $(\bar{C}^i(O^i))_{i\in I}$ is a core outcome.*

Proof. See Appendix A. $\square$

Intuitively, a fixed point of the generalized Gale-Shapley operator is in the core as, if any blocking contract $x \in X$ existed, then for each role χ in x, the lowest utility agents of the associated type $\mathsf{a}(\chi)$ would choose χ if it was available from their current set of available roles. But if this is true for every role associated with x, then $(O^i)_{i\in I}$ is not a fixed point as $\Xi_\chi^{\mathsf{a}(\chi)}((O^i)_{i\in I}) > O_\chi^{\mathsf{a}(\chi)}$ for each role χ associated with x by the definition of the operator Ξ.

5.3 Large Finite Economies

The core existence result for a continuum economy implies an approximate existence result for large finite economies, as in the case of stable outcomes.

Define a *finite economy* as a non-negative integer vector $n = (n^i)_{i\in I}$ specifying the number of agents of each type. The number of agents in economy n is denoted by $|n|$. A k-replica of economy n is denoted $k \cdot n$. A vector $(m_\chi^i)_{i\in I,\chi\in X_i}$ is a *core outcome of a finite economy* if it is a core outcome of the continuum model with $\theta = n$ where each coordinate of m is an integer. A finite economy n has a *core outcome excluding α agents* if there exists a finite economy $\bar{n}$ with a core outcome such that $\bar{n}^i \leq n^i$ for all $i \in I$ and $|n| \leq |\bar{n}| + \alpha$.

Proposition 2. *For every finite economy n there exist positive integers α and β such that:*

1. *Any replica of n has a core outcome excluding α agents.*
2. *For any k that is an integer multiple of β, the k-replica of n has a core outcome.*

Proof. See Appendix A. $\square$

Proposition 2 guarantees that, in a large replica economy, there is always an allocation that is an approximate core outcome. Intuitively, since an economy with a continuum of agents has a core outcome, it is possible to arrange most agents in any large finite replica into this outcome with only a bounded number of agents being assigned to different bundles of contracts.

6 Competitive Equilibria in Large Economies

6.1 Framework

We now consider the setting of Hatfield et al. (2013), where agents have quasilinear utility with respect to a numeraire commodity in ample supply. There is a set of *agent types* I, and a finite set of *trades* Ω. An agent of type $i \in I$ is endowed with the *valuation function* $u^i(\Phi, \Psi)$, where $\Phi \subseteq \Omega$ represents the trades for which agent i is a buyer, and $\Psi \subseteq \Omega$ represents the trades for which agent i is a seller. We allow $u^i(\Phi, \Psi)$ to take on any value in $[-\infty, \infty)$ for each $\Phi \subseteq \Omega$ and each $\Psi \subseteq \Omega$. We normalize the outside option as $u^i(\varnothing, \varnothing) = 0$ for each $i \in I$.[24]

A price vector $p \in \mathbb{R}^\Omega$ assigns a price p_ω for each trade $\omega \in \Omega$. Given a vector of prices p, define expenditure as the vector $e_p \in \mathbb{R}^{\mathcal{P}(\Omega)\times\mathcal{P}(\Omega)}$ such that

$$e_p(\Phi, \Psi) = \sum_{\varphi\in\Phi} p_\varphi - \sum_{\psi\in\Psi} p_\psi.$$

That is, $e_p(\Phi, \Psi)$ is the net transfer paid by an agent buying Φ and selling Ψ. Hence, the *utility* of a type i agent who buys contracts $\Phi \subseteq \Omega$ and sells contracts $\Psi \subseteq \Omega$ at prices p is given by

$$u^i(\Phi, \Psi) - e_p(\Phi, \Psi).$$

The set of agents is denoted I. An economy is given by a Lebesgue measurable distribution η over I, defined over a σ-algebra, and with $\eta(I) < \infty$. u^i is a measurable function of i.

[24]Note that, unlike in Hatfield et al. (2013), we allow here for any type of agent to buy (or sell) any contract. The constraint that a type cannot transact a contract is incorporated by setting the utility of buying (selling) to $-\infty$.

These conditions are satisfied if, for instance, utility functions are uniformly bounded.

An *allocation* is a measurable map

$$A : I \to \Delta(\mathcal{P}(\Omega) \times \mathcal{P}(\Omega))$$

specifying for each type $i \in I$ a distribution A^i over bundles of trades bought and sold. The space of allocations is denoted $\mathcal{A}$. Denote by $A^i(\Phi, \Psi)$ the proportion of agents of type i who buy the bundle of trades $\Phi \subseteq \Omega$ and sell the bundle of trades $\Psi \subseteq \Omega$.

Given an allocation A, we define the *excess demand* for each $i \in I$ and for each trade $\omega \in \Omega$ as

$$\mathsf{Z}^i_\omega(A) \equiv \sum_{\{\omega\} \subseteq \Phi \subseteq \Omega} \sum_{\Psi \subseteq \Omega} A^i(\Phi, \Psi) - \sum_{\{\omega\} \subseteq \Psi \subseteq \Omega} \sum_{\Phi \subseteq \Omega} A^i(\Phi, \Psi).$$

Define the *excess demand* for each trade $\omega \in \Omega$ for the entire economy as

$$\mathsf{Z}_\omega(A) \equiv \int_I \mathsf{Z}^i_\omega(A) \, \mathrm{d}\eta.$$

An allocation A is *feasible* if $\mathsf{Z}(A) = 0$. An *arrangement* $[A; p]$ is comprised of an allocation A and a price vector $p \in \mathbb{R}^\Omega$.

Definition 6. An arrangement $[A; p]$ is a *competitive equilibrium* if it satisfies two conditions:

1. Each agent obtains an optimal bundle given prices p, i.e., for all $i \in I$, $A^i(\Phi, \Psi) > 0$ only if

$$(\Phi, \Psi) \in \underset{(\tilde{\Phi}, \tilde{\Psi}) \in \mathcal{P}(\Omega) \times \mathcal{P}(\Omega)}{\arg\max} u^i(\tilde{\Phi}, \tilde{\Psi}) - e_p(\tilde{\Phi}, \tilde{\Psi}).$$

   If this is the case we say that A is *incentive compatible* given p.

2. A is a feasible allocation, i.e., $\mathsf{Z}(A) = 0$.

This is the standard notion of competitive equilibrium: the first condition ensures that each agent is optimizing given the prices p, and the second condition ensures that markets clear.

Finally, we will require some technical conditions in order to ensure the existence of a competitive equilibrium. An economy is *regular* if

1. The integral of absolute values of utility is finite, as long as agents are not given bundles for which they have utility of $-\infty$. That is,

$$\int_I \max_{\Phi,\Psi\subseteq\Omega, u^i(\Phi,\Psi)\neq-\infty} |u^i(\Phi,\Psi)| \,\mathrm{d}\eta < \infty.$$

2. Agents can supply any sufficiently small net demand for trades.[25] That is,

$$\{\mathsf{Y} \in \mathbb{R}^\Omega : \exists A \in \mathcal{A} \text{ such that } \mathsf{Y} = \mathsf{Z}(A) \text{ and } u^i(\Phi,\Psi) = -\infty \Rightarrow A^i(\Phi,\Psi) = 0\}$$

contains a neighborhood of 0.

6.2 Existence

We now establish existence of an equilibrium. Our proof strategy is to first show that there exists an allocation that maximizes total surplus. We then show that a surplus maximizing allocation is an equilibrium, when coupled with a vector of prices equal to the marginal social values of increasing the supply of each trade.

Theorem 3. *Every regular economy has a competitive equilibrium.*

Given prices p and an allocation A, denote the average utility received and prices paid by agents of type i as

$$u^i \cdot A^i \equiv \sum_{\Phi,\Psi\subseteq\Omega} u^i(\Phi,\Psi) \cdot A^i(\Phi,\Psi)$$

$$e_p \cdot A^i \equiv \sum_{\Phi,\Psi\subseteq\Omega} e_p(\Phi,\Psi) \cdot A^i(\Phi,\Psi)$$

[25] This condition rules out the case where there are no agents willing to sell a trade but demand is positive at any finite price. For example, if there is only one trade, $\Omega = \{\omega\}$, all agents have utility $-\infty$ when they are net sellers, and there are agents with arbitrarily high utility from being a net buyer, there is no equilibrium. The assumption rules out this example and similar cases involving sets of trades.

To prove the theorem, we introduce the social welfare function $\mathbf{W}(q)$, which denotes the maximal social welfare that may be attained by an allocation A such that $\mathsf{Z}(A) = q$. Formally,

$$\mathbf{W}(q) \equiv \sup_{\{A \in \mathcal{A} : \mathsf{Z}(A) = q\}} \int_I u^i \cdot A^i \, \mathrm{d}\eta.$$

$\mathbf{W}(q)$ attains its supremum as the argument of the supremum is a continuous function and the supremum is taken over a compact space for a suitably defined topology. We note this as a claim.

Claim 1. $\mathbf{W}(q)$ *attains its supremum. Formally,*

$$\mathbf{W}(q) = \max_{\{A \in \mathcal{A} : \mathsf{Z}(A) = q\}} \int_I u^i \cdot A^i \, \mathrm{d}\eta.$$

Proof. See Appendix A. □

The social welfare function also satisfies the following properties.

1. $\mathbf{W}$ is uniformly bounded above:

$$\mathbf{W}(q) \leq \int_I \max_{\Phi, \Psi \subseteq \Omega} u^i(\Phi, \Psi) \, \mathrm{d}\eta$$

   for any $q \in \mathbb{R}^\Omega$, and this latter quantity is finite in a regular economy.

2. $\mathbf{W}(q) > -\infty$ for all q in a neighborhood of 0: By parts 1 and 2 of the definition of a regular economy, for any vector q with small enough norm there are enough agents to absorb the excess of any trades while incurring only finite disutility, and hence $\mathbf{W}(q) > -\infty$.

3. $\mathbf{W}$ is concave: Consider any two stocks q and $\tilde{q}$ in $\mathbb{R}^\Omega$, and let

$$\begin{aligned} A &\in \operatorname*{arg\,max}_{\hat{A}\in\{\dot{A}\in\mathcal{A}:\mathsf{Z}(\dot{A})=q\}} \int_I u^i \cdot \hat{A}^i \,\mathrm{d}\eta \\ \tilde{A} &\in \operatorname*{arg\,max}_{\hat{A}\in\{\dot{A}\in\mathcal{A}:\mathsf{Z}(\dot{A})=\tilde{q}\}} \int_I u^i \cdot \hat{A}^i \,\mathrm{d}\eta. \end{aligned}$$

For each $\alpha \in [0,1]$, we have that $\mathsf{Z}(\alpha A + (1-\alpha)\tilde{A}) = \alpha q + (1-\alpha)\tilde{q} \equiv \bar{q}$. Hence, letting $\bar{A} \equiv (\alpha A^i + (1-\alpha)\tilde{A}^i)$, we have that

$$\begin{aligned} \mathbf{W}(\bar{q}) &\geq \int_I u^i \cdot \bar{A}^i \,\mathrm{d}\eta \\ &= \int_I u^i \cdot (\alpha A^i + (1-\alpha)\tilde{A}^i) \,\mathrm{d}\eta \\ &= \alpha \int_I u^i \cdot A^i \,\mathrm{d}\eta + (1-\alpha) \int_I u^i \cdot \tilde{A}^i \,\mathrm{d}\eta \\ &= \alpha \mathbf{W}(q) + (1-\alpha)\mathbf{W}(\tilde{q}). \end{aligned}$$

These properties imply that $\mathbf{W}(0)$ attains a maximum at some allocation A. Moreover, there exists at least one supergradient p of $\mathbf{W}$ at $q = 0$.[26] The arrangement $[A; p]$ satisfies the market clearing condition of Definition 6, that is, $\mathsf{Z}(A) = 0$. Let J be the set of types such that some agents of that type do not get optimal bundles given prices p.

We will now show that $\eta(J) = 0$. To demonstrate this, we will show that $\eta(J) > 0$ leads to a contradiction. Let $\tilde{A}$ be an allocation such that $\tilde{A}^i = A^i$ for $I \smallsetminus J$ and $\tilde{A}^i$ is a distribution over optimal bundles given p otherwise, and let $\tilde{q} = \mathsf{Z}(\tilde{A})$. We have

$$\begin{aligned} \mathbf{W}(\tilde{q}) &\geq \int_I u^i \cdot \tilde{A}^i \,\mathrm{d}\eta \\ &> \int_I u^i \cdot A^i - (e_p \cdot A^i - e_p \cdot \tilde{A}^i) \,\mathrm{d}\eta \\ &= \mathbf{W}(0) - p \cdot \mathsf{Z}(A) + p \cdot \mathsf{Z}(\tilde{A}) = \mathbf{W}(0) + p \cdot \tilde{q}. \end{aligned}$$

[26] This follows from Theorem 23.4 of Rockafellar (1970) because $\mathbf{W}$ is concave, and (1) and (2) imply that $\mathbf{W}$ is proper and that 0 is in the relative interior of the domain of $\mathbf{W}$.

The first inequality follows from the definition of $\mathbf{W}$. The second inequality from the fact that all agents prefer to buy $\tilde{A}$ to A, and the strictness of the inequality follows as $\eta(J) > 0$. In the third line, the first equality follows from the optimality of A and the second equality follows from the definition of $\tilde{q}$. The result that $\mathbf{W}(\tilde{q}) > \mathbf{W}(0) + p \cdot \tilde{q}$ contradicts the fact that p is a supergradient. The contradiction implies that $\eta(J) = 0$.

To complete the proof of Theorem 3, note that $\eta(J) = 0$ implies that $\mathsf{Z}(\tilde{A}) = 0$. Moreover, $\tilde{A}$ is incentive compatible given p by definition. Therefore, $[\tilde{A}; p]$ is an equilibrium.

Our setting is related to models of general equilibrium with indivisible commodities and transferable utility. In settings with a finite number of agents, a number of papers (Gul and Stacchetti, 1999, 2000; Sun and Yang, 2006, 2009; Hatfield et al., 2013) show the existence of competitive equilibrium under the assumption that preferences are substitutable or other restrictions on preferences (Baldwin and Klemperer, 2015).[27] We do not impose substantive restrictions on preferences but instead assume that the set of agents is a continuum. The most closely related work is by Azevedo et al. (2013), who prove the existence of competitive equilibria in the setting of Gul and Stacchetti (1999) in a model with a continuum of agents. We generalize their result by allowing for relationship-specific utility and for the assumption that some agents cannot engage in some trades (as in our model utility may take on the value $-\infty$). Hence, our results require a proof technique that is quite different from that of Azevedo et al. (2013), who employ a fixed point argument. Their argument does not work in our setting because the tâtonnement process they consider does not take bounded sets into bounded sets. This occurs due to the possibility of $-\infty$ utility for some

[27] Baldwin and Klemperer (2015) approach this problem with a novel method by using techniques from tropical geometry. They give necessary and sufficient conditions on preferences for the existence of equilibria that are different than the substitutes conditions in the literature. It is easier to understand their approach with a simple example, which we take from Azevedo et al. (2013). Consider a setting with two indivisible goods where a single unit of each good available. Sonia is willing to pay \$75 for either good (or both). Charlie is willing to pay \$100 for both goods but has no value for a single item. No competitive equilibrium exists. Most of the literature would attribute the non-existence of equilibria to the fact that Charlie views the goods as complements. Baldwin and Klemperer, by contrast, note that as a price changes Sonia's demand can change in the directions $(1,0)$, $(0,1)$, and $(1,-1)$, while Charlie's demand can only change in the direction $(1,1)$. Baldwin and Klemperer demonstrate that the fact that not all matrices formed by pairs of these vectors have determinant 0 or ± 1 implies that this class of demand functions can preclude existence of equilibrium.

bundles, which means that even at very high prices there may be excess demand for some trades. Instead, our proof is based on constructing an equilibrium from a welfare-maximizing allocation, an idea pioneered by Gretsky et al. (1992, 1999) for the continuum assignment problem.

6.3 Efficiency and Uniqueness

A feasible allocation A is *efficient* if A maximizes welfare. That is, if, for any feasible allocation $\tilde{A}$,

$$\int_I u^i \cdot A^i \,\mathrm{d}\eta \geq \int_I u^i \cdot \tilde{A}^i \,\mathrm{d}\eta. \tag{5}$$

A competitive equilibrium $[A; p]$ is *efficient* if A is efficient.

Proposition 3. *Every competitive equilibrium is efficient.*

Proof. See Appendix A. □

Economies with sufficiently rich preference heterogeneity have a unique equilibrium price vector. To state this result we define the following notion of preference heterogeneity.

Definition 7. The distribution η has full support if, for every open set $U \subseteq \mathbb{R}^{\mathcal{P}(\Omega)\times\mathcal{P}(\Omega)}$ we have $\eta(I^U) > 0$, where

$$I^U = \{i \in I : \text{the vector } u^i \in U\}.$$

We can now state the uniqueness result.

Proposition 4. *A regular economy where η has full support has a unique vector of competitive equilibrium prices.*

Proof. See Appendix A. □

This result is analogous to the uniqueness result in Azevedo et al. (2013). Intuitively, in a market with sufficiently rich preferences, in equilibrium there are always agents who are close to indifferent between engaging in a contract or not, and some of these marginal agents are engaging in the contract, and some are not. This implies that, if external agents were to supply or demand a small quantity of this contract, the gain or loss in social welfare would be proportional to the equilibrium price. Mathematically, this implies that the function $\mathbf{W}$ is differentiable at 0, and therefore that $\mathbf{W}$ has a unique supergradient. The fact that every equilibrium price is a supergradient implies that equilibrium prices are unique.

6.4 Large Finite Economies

A *finite economy* is defined as a vector $n \equiv (n^i)_{i \in I}$ specifying the number of agents of each type i. Each $n^i \in \mathbb{Z}_{\geq 0}$ and $n^i = 0$ for all but a finite set of types. The total number of agents in the finite economy n is denoted $|n|$. Given a natural number k, the *k-replica* of the finite economy n is the finite economy $k \cdot n$, which has k copies of each agent present in n. The *∞-replica* is the continuum economy η_n given by

$$\eta_n = \sum_i \frac{n^i}{|n|} \cdot \delta^i,$$

where δ^i is Dirac delta function placing mass 1 on i. A finite economy is *regular* if its ∞-replica is regular.

An *allocation* of a finite economy n is an allocation A of the ∞-replica such that, for all i with $n^i \neq 0$, the coordinates of A^i are integer multiples of $\frac{1}{n^i}$. A *competitive equilibrium* of a finite economy is a pair $[A; p]$ such that A is an allocation of the finite economy and $[A; p]$ is an equilibrium of the ∞-replica. We say that a finite economy n has a *competitive equilibrium excluding α agents* if there exists a finite economy $\bar{n}$ with a competitive equilibrium such that $\bar{n}^i \leq n^i$ for all $i \in I$ and $|n| \leq |\bar{n}| + \alpha$. As in the earlier sections, the intuition is that it is possible to reach a competitive equilibrium by selecting α agents and either excluding them from trade or assigning them non-optimal, but individually rational, bundles.

The following proposition establishes an approximate existence result that is similar to that in the non-quasilinear case.

Proposition 5. *Consider a regular finite economy n. There exist positive integers α and β such that:*

1. *Any replica of n has a competitive equilibrium excluding α agents.*

2. *For any k that is an integer multiple of β, the k-replica of n has a competitive equilibrium.*

Proof. See Appendix A. □

7 Conclusion

Complementarities are an important feature of many matching markets. However, models of matching with heterogeneous preferences have had trouble incorporating complementarities as complementarities can preclude the existence of equilibrium in markets with a finite number of traders.

In this work we have asked whether complementarities are still an issue in large markets, formalized as markets with a continuum of traders. Our results show that stable outcomes may still fail to exist even in a market with a continuum of traders. Therefore, the non-existence of stable outcomes in models with a finite number of agents is not simply due to the finiteness of these markets. However, our results also show that equilibrium exists in large markets much more generally than in markets with a finite number of agents.

In particular, we show that in settings that do incorporate complementarities, such as buyer-seller markets with substitutable preferences on one side and network economies with transferable utility, equilibria do exist. Finally, it is also shown that even when stable outcomes do not exist, the core of a large matching market is always nonempty. We also demonstrate the existence of approximately stable matchings in large finite markets (with the caveat that whether approximate stability is appropriate as a solution concept or mechanism design goal depends on the specific setting).

This work points to some interesting directions for future work. First, it allows for the modeling of economies where complementarities and heterogeneous preferences are present.

Second, extensions of standard empirical models of matching markets such as Choo and Siow (2006) that incorporate complementarities always have equilibria, so that these phenomena may be investigated empirically. Third, despite possible non-existence of stable outcomes in labor market clearinghouses with couples, it is possible to design mechanisms that always produce an approximately stable outcome.

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A Proofs

A.1 Proof of Lemma 1

We first show that $(((h^b(O^B))_{b\in B}, (h^s(O^S))_{s\in S})$ is an outcome. Condition 1 of Definition 2 is satisfied as, for each $b \in B$, $h^b_{\varnothing}(O^B) = \theta^b - \sum_{k=1}^K h^b_{Y^k}(O^B)$, and for each $s \in S$, $h^s_{\varnothing}(O^S) = \theta^s - \sum_{k=1}^K h^s_{Y^k}(O^S)$ by Eq. (1). To see that Condition 2 of Definition 2 is satisfied, suppose that $h^{\mathsf{s}(x)}_x(O^S) < h^{\mathsf{b}(x)}_x(O^B)$ for some $x \in X$.[28] There are two cases:

[28]The case where $h^{\mathsf{s}(x)}_x(O^S) > h^{\mathsf{b}(x)}_x(O^B)$ is analogous.

1. $h_x^{\mathsf{s}(x)}(O^S) = \bar{C}_x^{\mathsf{s}(x)}((O^S_{X \smallsetminus \{x\}}, \theta^{\mathsf{s}(x)}))$. Then $O_x^B = h_x^{\mathsf{s}(x)}(O^S)$ by Eq. (2), hence $h_x^{\mathsf{b}(x)}(O^B) \leq O_x^B = h^{\mathsf{s}(x)}(O^S)$, a contradiction.

2. $h_x^{\mathsf{s}(x)}(O^S) < \bar{C}_x^{\mathsf{s}(x)}((O^S_{X \smallsetminus \{x\}}, \theta^{\mathsf{s}(x)}))$. This implies by Eq. (1) that $h_x^{\mathsf{s}(x)}(O^S) = O_x^S$. But, by Eq. (2), $O_x^S = \bar{C}_x^{\mathsf{b}(x)}((O^B_{X \smallsetminus \{x\}}, \theta^{\mathsf{b}(x)})) \geq h_x^{\mathsf{b}(x)}(O^B)$, which implies that $h_x^{\mathsf{b}(x)}(O^B) \leq O_x^S = h^{\mathsf{s}(x)}(O^S)$, a contradiction.

We now show that the outcome $((h^b(O^B))_{b \in B}, (h^s(O^S))_{s \in S})$ is stable. It is immediate that it is individually rational by the definitions of $\bar{C}^b$ and $\bar{C}^s$. Suppose that there exists a blocking multiset Z (and associated partitions $\{Z^{\mathfrak{b}}\}_{\mathfrak{b} \in \mathfrak{B}}$ and $\{Z^{\mathfrak{s}}\}_{\mathfrak{s} \in \mathfrak{S}}$, along with associated sets $\{Y^{\mathfrak{b}}\}_{\mathfrak{b} \in \mathfrak{B}}$ and $\{Y^{\mathfrak{s}}\}_{\mathfrak{s} \in \mathfrak{S}}$). Since the preferences of each buyer are substitutable, if $z \in Z^{\mathfrak{b}}$, then $z \in C^{\mathsf{b}(z)}(\{z\} \cup Y^{\mathfrak{b}})$. Hence, $z \in C^{\mathsf{b}(z)}(\{z\} \cup Y^{\mathfrak{b}})$ for each $z \in Z$. Hence, by Eq. (2), it must be that $O_z^S = \Phi_z^S(O^B) > h_z^{\mathsf{b}(z)}(O^B) = h_z^{\mathsf{s}(z)}(O^S)$ for all $z \in Z$ (where the equalities follow as (O^B, O^S) is a fixed point). But then any $Z^{\mathfrak{s}}$ can be chosen by the corresponding seller s, and so (O^B, O^S) is not a fixed point.

A.2 Proof of Proposition 1

Consider the continuum model with $\theta = n$. By Theorem 1 the continuum model has a stable outcome $\bar{m}$. Consider the set M^* of all outcome vectors m in the continuum model such that the support of m is contained in the support of $\bar{m}$. Note that, by the definition of stability, every outcome in M^* is stable. The set M^* can be written as the set of all vectors $(m_Z^i)_{i \in I, Z \in \mathcal{P}(X_i)}$ such that

$$
\begin{aligned}
& m \geq 0, \\
& \sum_{\{x\}\subseteq Z \subseteq X_{\mathsf{s}(x)}} m_Z^{\mathsf{s}(x)} = \sum_{\{x\}\subseteq Z \subseteq X_{\mathsf{b}(x)}} m_Z^{\mathsf{b}(x)} \text{ for all } x, \\
& \sum_{Z \in \mathcal{P}(X_i)} m_Z^i = \theta^i \text{ for all } i, \text{ and} \\
& m_Z^i = 0 \text{ if } \bar{m}_Z^i = 0 \text{ for all } i \text{ and } Z.
\end{aligned}
$$

Because M^* is a bounded and non-empty polytope, it has an extreme point m^*. Theorem 2.3 of Bertsimas and Tsitsiklis (1997) implies that the extreme point m^* is a basic feasible solution to these constraints. Theorem 2.2 of Bertsimas and Tsitsiklis (1997) implies that this basic feasible solution can be written as the product of the inverse of a matrix with integer entries and an integer vector. By Cramer's rule, all the entries of m^* are rational numbers. Consequently, there exists an integer β such that $k\beta \cdot m^*$ is an integer vector for all integer k. Moreover, because m^* is a stable outcome of the continuum model, $k\beta \cdot m^*$ is a stable outcome of the finite economy $k\beta \cdot n$. This proves the second part of the proposition.

As for the first part of the proposition, consider an arbitrary replica $k \cdot n$. Let k' be the smallest multiple of β that is no greater than k. The economy $k' \cdot n$ has a stable outcome, by part 2 of the proposition. Moreover, $k' \cdot n$ only excludes $(k - k')|n| \leq (\beta - 1)|n|$ agents. This establishes part 1 of the proposition taking $\alpha = (\beta - 1)|n|$.

A.3 Proof of Lemma 2

We first show that if $\Xi((O^i)_{i\in I}) = (O^i)_{i\in I}$, then $(\bar{C}^i(O^i))_{i\in I}$ is an outcome. It is immediate from the definition of the choice operator (3) that $(\bar{C}^i(O^i))_{i\in I}$ satisfies Condition 1 of Definition 4. Suppose $(\bar{C}^i(O^i))_{i\in I}$ does not satisfy Condition 2 of Definition 4; then there exists a contract

x and roles $y, z \in x$ such that $\bar{C}_y^{\mathsf{a}(y)}(O^{\mathsf{a}(y)}) < \bar{C}_z^{\mathsf{a}(z)}(O^{\mathsf{a}(z)})$. Let

$$\mathcal{Y} \equiv \arg\min_{y \in x} \bar{C}_y^{\mathsf{a}(y)}(O^j);$$

note that there exists a role $z \in x \smallsetminus \mathcal{Y}$, i.e., a role z such that $\bar{C}_z^{\mathsf{a}(z)}(O^{\mathsf{a}(z)}) > \bar{C}_y^{\mathsf{a}(y)}(O^{\mathsf{a}(y)})$ for each $y \in \mathcal{Y}$. There are two cases:

1. For some $y \in \mathcal{Y}$, $\bar{C}_y^{\mathsf{a}(y)}(O^{\mathsf{a}(y)}_{X_{\mathsf{a}(y)} \smallsetminus \{y\}}, \theta^{\mathsf{a}(y)}) \leq O_y^{\mathsf{a}(y)}$. Then

   $$\bar{C}_y^{\mathsf{a}(y)}(O^{\mathsf{a}(y)}_{X_{\mathsf{a}(y)} \smallsetminus \{y\}}, \theta^{\mathsf{a}(y)}) = \bar{C}_y^{\mathsf{a}(y)}(O^{\mathsf{a}(y)})$$

   by the definition of the choice operator (3). However,

   $$O_z^{\mathsf{a}(z)} = \Xi_z^{\mathsf{a}(z)}((O^i)_{i \in I}) \leq \bar{C}_y^{\mathsf{a}(y)}(O^{\mathsf{a}(y)}_{X_{\mathsf{a}(y)} \smallsetminus \{y\}}, \theta^{\mathsf{a}(y)})$$

   by Eq. (4). Combining these expressions, $O_z^{\mathsf{a}(z)} \leq \bar{C}_y^{\mathsf{a}(y)}(O^{\mathsf{a}(y)})$. Hence, $\bar{C}_z^{\mathsf{a}(z)}(O^{\mathsf{a}(z)}) \leq \bar{C}_y^{\mathsf{a}(y)}(O^{\mathsf{a}(y)})$; but this contradicts the assumption that $z \notin x \smallsetminus \mathcal{Y}$.

2. For all $y \in \mathcal{Y}$, $\bar{C}_y^{\mathsf{a}(y)}(O^{\mathsf{a}(y)}_{X_{\mathsf{a}(y)} \smallsetminus \{y\}}, \theta^{\mathsf{a}(y)}) > O_y^{\mathsf{a}(y)}$. Then $O_y^{\mathsf{a}(y)} = \bar{C}_y^{\mathsf{a}(y)}(O^{\mathsf{a}(y)})$ by the definition of the choice operator (3), and so $O_y^{\mathsf{a}(y)} = O_{\hat{y}}^{\mathsf{a}(\hat{y})}$ for all $y, \hat{y} \in \mathcal{Y}$. Furthermore, since for all $z \in x \smallsetminus \mathcal{Y}$, $\bar{C}_z^{\mathsf{a}(z)}(O^{\mathsf{a}(z)}) > \bar{C}_y^{\mathsf{a}(y)}(O^{\mathsf{a}(y)})$ for each $y \in \mathcal{Y}$, we have that $\bar{C}_z^{\mathsf{a}(z)}(O^{\mathsf{a}(z)}_{X_{\mathsf{a}(z)} \smallsetminus \{z\}}, \theta^{\mathsf{a}(z)}) > \bar{C}_y^{\mathsf{a}(y)}(O^{\mathsf{a}(y)}) = O^{\mathsf{a}(y)}$ for each $y \in \mathcal{Y}$. Hence,

   $$\min_{\chi \in x} \bar{C}_y^{\mathsf{a}(\chi)}((O^{\mathsf{a}(\chi)}_{X_{\mathsf{a}(\chi)} \smallsetminus \{\chi\}}, \theta^{\mathsf{a}(\chi)})) > O^{\mathsf{a}(y)}$$

   for each $y \in \mathcal{Y}$, and so $(O^i)_{i \in I}$ cannot be a fixed point.

Hence, for any fixed point $(O^i)_{i \in I}$, $(O^i)_{i \in I}$ is an outcome.

We now show that if $\Xi((O^i)_{i \in I}) = (O^i)_{i \in I}$, then $(\bar{C}^i(O^i))_{i \in I}$ is a core outcome. Suppose not; then there exists a contract $x \in X$ such that for all $y \in x$, there exists $z \in X_{\mathsf{a}(y)}$ such that $y \succ_{\mathsf{a}(y)} z$ and $\bar{C}_z^{\mathsf{a}(y)}(O^{\mathsf{a}(y)}) > 0$. Hence, by the definition of the choice operator (3), $\bar{C}_y^{\mathsf{a}(y)}((O^{\mathsf{a}(y)}_{X_{\mathsf{a}(y)} \setminus \{y\}}, \theta^{\mathsf{a}(y)})) > O^{\mathsf{a}(y)}$ for each $y \in x$, and so $(O^i)_{i \in I}$ cannot be a fixed point.

A.4 Proof of Proposition 2

Consider the continuum model with $\theta = n$. By Theorem 2 the continuum model has a core outcome $\bar{m}$. Consider the set M^* of all outcome vectors m in the continuum model such that the support of m is contained in the support of $\bar{m}$. Note that, by the definition of the core, every outcome in M^* is in the core. The set M^* can be written as the set of all vectors $(m^i_\chi)_{i \in I, \chi \in X_i}$ such that

$$
\begin{aligned}
& m \geq 0, \\
& m_\chi^{\mathsf{a}(\chi)} = m_y^{\mathsf{a}(y)} \text{ for all } x \in X \text{, and all } \chi, y \in x, \\
& \sum_{\chi \in X_i} m^i_\chi = \theta^i \text{ for all } i \text{, and} \\
& m^i_\chi = 0 \text{ if } \bar{m}^i_\chi = 0 \text{ for all } i \text{ and } \chi.
\end{aligned}
$$

Because M^* is a bounded and non-empty polytope, it has an extreme point m^*. Theorem 2.3 of Bertsimas and Tsitsiklis (1997) implies that the extreme point m^* is a basic feasible solution to these constraints. Theorem 2.2 of Bertsimas and Tsitsiklis (1997) implies that this basic feasible solution can be written as the product of the inverse of a matrix with integer entries and an integer vector. By Cramer's rule, all the entries of m^* are rational numbers. Consequently, there exists an integer β such that $k\beta \cdot m^*$ is an integer vector for all integer k. Moreover, because m^* is a core outcome of the continuum model, $k\beta \cdot m^*$ is a core outcome of the finite economy $k\beta \cdot n$. This proves the second part of the proposition.

The first part of the proposition then follows from essentially the same argument as Proposition 1.

A.5 Proof of Claim 1

Let

$$\tilde{\mathcal{A}} = \{A \in \mathcal{A} : \mathsf{Z}(A) = q, \text{ and } u^i(\Phi, \Psi) = -\infty \implies A^i(\Phi, \Psi) = 0\}.$$

If $\mathbf{W}(q) = -\infty$, the claim is trivial. Consider the case where $\mathbf{W}(q) > -\infty$. The definition of $\mathbf{W}(q)$ implies that it is sufficient to take the supremum in $\tilde{\mathcal{A}}$. This set is compact in the product topology (pointwise convergence), by Tychonoff's Theorem. Consider a sequence of allocations A_k converging pointwise to an allocation A_∞. We can show that

$$\lim_{k \to \infty} \int_I u^i \cdot A_k^i \, \mathrm{d}\eta = \int_I u^i \cdot A_\infty^i \, \mathrm{d}\eta.$$

We have that $|u^i \cdot A^i|$ is bounded by $\max_{\Phi,\Psi} u^i(\Phi, \Psi)$. Moreover, regularity implies that

$$\int_I \max_{\Phi,\Psi} u^i(\Phi, \Psi) \, \mathrm{d}\eta$$

is finite. Convergence of the desired integrals then follows from the dominated convergence theorem.

The convergence above implies that

$$\int_I u^i \cdot A^i \, \mathrm{d}\eta$$

varies continuously with A in the compact set $\tilde{\mathcal{A}}$. Therefore, the supremum in the definition of $\mathbf{W}(q)$ attains its maximum.

A.6 Proof of Proposition 3

Consider a competitive equilibrium $[A; p]$ and any feasible allocation $\tilde{A}$. Individual optimization (Condition 1 of Definition 6) implies that, for all $i \in I$,

$$(u^i - e_p) \cdot A^i \geq (u^i - e_p) \cdot \tilde{A}^i.$$

Integrating this, we have that

$$\int_I (u^i - e_p) \cdot A^i \,\mathrm{d}\eta \geq \int_I (u^i - e_p) \cdot \tilde{A}^i \,\mathrm{d}\eta.$$

We have that $\mathsf{Z}(A) = \mathsf{Z}(\tilde{A}) = 0$ because both allocations are feasible. Hence, $\int_I e_p \cdot A^i \,\mathrm{d}\eta = \int_I e_p \cdot \tilde{A}^i \,\mathrm{d}\eta = 0$. Therefore, the above inequality is equivalent to Eq. (5), completing the proof.

A.7 Proof of Proposition 4

We first prove the following Lemma.

Lemma 3. *Every equilibrium price vector is a supergradient of* $\mathbf{W}$ *at* 0.

Proof. Consider an equilibrium $[A; p]$, and a vector $q \in \mathbb{R}^\Omega$. Let $\tilde{A}$ be an allocation with $\mathsf{Z}(\tilde{A}) = q$. Individual optimization (Condition 1 of Definition 6) implies that, for each $i \in I$,

$$(u^i - e_p) \cdot A^i \geq (u^i - e_p) \cdot \tilde{A}^i.$$

Integrating this we have

$$\int_I (u^i - e_p) \cdot A^i \,\mathrm{d}\eta \geq \int_I (u^i - e_p) \cdot \tilde{A}^i \,\mathrm{d}\eta.$$

Therefore,

$$\int_I u^i \cdot \tilde{A}^i \, \mathrm{d}\eta \le \int_I u^i \cdot A^i \, \mathrm{d}\eta + \int_I e_p \cdot (\tilde{A}^i - A^i) \, \mathrm{d}\eta$$
$$\int_I u^i \cdot \tilde{A}^i \, \mathrm{d}\eta \le \mathbf{W}(0) + p \cdot q.$$

The inequality holds for any such $\tilde{A}$. This implies that $\mathbf{W}(q) \le \mathbf{W}(0) + p \cdot q$, completing the proof. □

We now prove Proposition 4. Consider an equilibrium $[A; p]$. Fix a trade ω and $\epsilon > 0$. Define the marginal non-buyers of trade ω as the agent types i who do not buy trade ω at p, but who would gain utility of at least $p_\omega - \epsilon$ by adding trade ω to their bundle. Formally,

$$M(\epsilon) \equiv \{i \in I : A^i(\Phi, \Psi) > 0 \implies \omega \notin \Phi, \omega \notin \Psi, \text{ and } \\ u^i(\Phi \cup \{\omega\}, \Psi) - u^i(\Phi, \Psi) > p_\omega - \epsilon\}.$$

By the full support assumption, $M(\epsilon)$ has positive measure. Consider a vector $q \in \mathbb{R}^\Omega$ such that $q_\omega = \delta > 0$ and $q_\psi = 0$ for all $\psi \in \Omega \smallsetminus \{\omega\}$. For δ small enough, there exists an allocation $\tilde{A}$ with $\mathsf{Z}(\tilde{A}) = q$ such that $\tilde{A}^i = A^i$ for all $i \in I \smallsetminus M(\epsilon)$ and that assigns the extra mass δ of trade ω to marginal non-buyers in the set $M(\epsilon)$. Therefore, by the definition of $M(\epsilon)$, we have that

$$\mathbf{W}(q) - \mathbf{W}(0) \ge \delta \cdot (p_\omega - \epsilon).$$

By Lemma 3 the price vector p is a supergradient, which implies that

$$p_\omega - \epsilon \le \frac{\mathbf{W}(q) - \mathbf{W}(0)}{\delta} \le p_\omega.$$

Moreover, the inequalities hold for all q with sufficiently small norm because can make an analogous argument for $q_\omega = \delta < 0$. Therefore, $\mathbf{W}$ has a directional derivative at 0, and $\partial_\omega \mathbf{W}(0) = p_\omega$. The fact that this directional derivative is well-defined implies that equilibrium prices are unique and equal to the marginal social value of each trade ω.

A.8 Proof of Proposition 5

The ∞-replica η_n is regular, and therefore has an equilibrium $[\bar{A}; p^*]$.

Consider the set $\mathcal{A}^*$ of all feasible allocations A such that, for all i with $n^i = 0$, $A^i = \bar{A}^i$, and for all i with $n^i > 0$ the support of A^i is contained in the support of $\bar{A}^i$. Note that $[A; p^*]$ is a competitive equilibrium of the ∞-replica for any A in the set $\mathcal{A}^*$.

Moreover, the bundles of contracts that agents with $n^i > 0$ receive in the allocations in $\mathcal{A}^*$,

$$\cup_{A \in \mathcal{A}^*} (A^i)_{i:n^i>0},$$

is the set of vectors $(A^i)_{i:n^i>0}$ that solve

$$\begin{aligned}
&A^i(\Phi, \Psi) \geq 0 \text{ for all } i, \Phi, \text{ and } \Psi, \\
&\sum_{i,\Phi,\Psi:\omega\in\Phi} A^i(\Phi, \Psi) \cdot n^i = \sum_{i,\Phi,\Psi:\omega\in\Psi} A^i(\Phi, \Psi) \cdot n^i \text{ for all } x, \\
&\sum_{\Phi,\Psi} A^i(\Phi, \Psi) = 1 \text{ for all } i, \text{ and} \\
&A^i(\Phi, \Psi) = 0 \text{ if } \bar{A}^i(\Phi, \Psi) = 0.
\end{aligned}$$

Because $\mathcal{A}^*$ is a bounded and non-empty polytope, it has an extreme point $(A_i^*)_{i:n^i>0}$. Theorem 2.3 of Bertsimas and Tsitsiklis (1997) implies that the extreme point is a basic feasible solution to these constraints. Theorem 2.2 of Bertsimas and Tsitsiklis (1997) implies that this basic feasible solution can be written as the product of the inverse of a matrix with integer entries and an integer vector. By Cramer's rule, all the entries of $(A_i^*)_{i:n^i>0}$ are rational numbers. Thus, there exists an allocation A^* such that $[A^*; p]$ is an equilibrium of the ∞-replica, and all the coordinates of A^* with $n^i > 0$ are rational numbers.

Consequently, there exists an integer β such that all the coordinates of A^* are integer multiples of $1/\beta$. Therefore, $[A^*; p]$ is a competitive equilibrium of any k-replica where k is a multiple of β.

The first part of the proposition then follows from essentially the same argument as Proposition 1.

B Reduction of a Multiunit Demand Economy to a Unit Demand Economy

In this appendix, we consider a model where each agent may demand multiple contracts. We show that this model may be regarded as equivalent to the model of Section 5, where each agent demands a single contract. In order to distinguish the notion of contract in the two models, we refer to contracts in the single unit demand economy as *übercontracts* (and roles of the single unit demand economy as *überroles*.) In particular, we show that any core allocation of a unit demand economy induced by a multiunit demand economy corresponds to a core allocation in the original multiunit demand economy.

B.1 Framework of the Multiunit Demand Economy

There exists a finite set of *roles* $\mathfrak{X}$, and each role $\mathfrak{x} \in \mathfrak{X}$ is identified with an agent type $\mathsf{a}(\mathfrak{x}) \in I$. For $\mathfrak{Y} \subseteq \mathfrak{X}$, we let $\mathsf{a}(\mathfrak{Y}) \equiv \{i \in I : \exists \mathfrak{y} \in \mathfrak{Y} \text{ such that } i = \mathsf{a}(\mathfrak{y})\}$ and $\mathfrak{Y}_i \equiv \{\mathfrak{y} \in \mathfrak{Y} : i = \mathsf{a}(\mathfrak{y})\}$. A contract x is a set of roles, i.e., $\mathsf{x} \subseteq \mathfrak{X}$; we denote by X the set of all contracts. Furthermore, as in Section 5, each contract is composed of contract-specific roles, i.e., $\mathsf{x} \cap \mathsf{y} = \varnothing$.

Each agent type $i \in I$ is endowed with a weak preference $\trianglerighteq_i$ over subsets of the set of roles $\mathfrak{X}_i$: for sets of roles $\mathfrak{Y}, \mathfrak{Z} \subseteq \mathfrak{X}_i$, we say that $\mathfrak{Y} \rhd_i \mathfrak{Z}$ if $\mathfrak{Y}$ is strictly preferred to $\mathfrak{Z}$ and $\mathfrak{Y} \trianglerighteq_i \mathfrak{Z}$ if $\mathfrak{Y}$ is weakly preferred to $\mathfrak{Z}$. We naturally extend this preference relation to subsets of $\mathfrak{X}$: for $\mathfrak{Y}, \mathfrak{Z} \subseteq \mathfrak{X}$, we say $\mathfrak{Y} \rhd_i \mathfrak{Z}$ if $\mathfrak{Y}_i \rhd_i \mathfrak{Z}_i$ and $\mathfrak{Y} \trianglerighteq_i \mathfrak{Z}$ if $\mathfrak{Y}_i \trianglerighteq_i \mathfrak{Z}_i$.

Definition 8. An *outcome* is a vector $(m^i)_{i \in I}$, where $m^i \in [0, \theta^i]^{\mathcal{P}(\mathfrak{X}_i)}$, such that

1. for all $i \in I$, $\sum_{\mathfrak{Z} \subseteq \mathfrak{X}_i} m^i_{\mathfrak{Z}} = \theta^i$, and

2. for all $\mathsf{x} \in \mathsf{X}$, for all $\mathfrak{x}, \mathfrak{y} \in \mathsf{x}$, $\sum_{\{\mathfrak{x}\} \subseteq \mathfrak{Z} \subseteq \mathfrak{X}_i} m_{\mathfrak{Z}}^{\mathsf{a}(\mathfrak{x})} = \sum_{\{\mathfrak{y}\} \subseteq \mathfrak{Z} \subseteq \mathfrak{X}_i} m_{\mathfrak{Z}}^{\mathsf{a}(\mathfrak{y})}$.

The first condition ensures that each type of agent is fully assigned to some subset of roles (possibly the empty set, which denotes the outside option in this context). The second condition ensures that "supply meets demand", that is, for each contract, each role has an equal measure of agents (of the appropriate type) performing that role.

We now define the core in this more general economy.

Definition 9. An outcome $(m^i)_{i \in I} \in \times_{i \in I} [0, \theta^i]^{\mathcal{P}(\mathfrak{X}_i)}$ is in the *core* if

1. for each $i \in I$, for each $\mathfrak{Y} \subseteq \mathfrak{X}_i$, if $\varnothing \triangleright_i \mathfrak{Y}$, then $m^i_{\mathfrak{Y}} = 0$, and
2. there does not exist a set of contracts $\mathsf{Y} \subseteq \mathsf{X}$ such that, for each $i \in \mathsf{a}(\mathsf{Y})$, there exists a partition of Y_i into nonempty sets $\{\mathsf{Y}^i(n)\}_{n \in N(i)}$ and a set of roles $\mathfrak{Z}^i \subseteq \mathfrak{X}_i$ such that
   (a) $m^i_{\mathfrak{Z}^i} > 0$, and
   (b) for each $n \in N(i)$, $\cup_{\mathsf{y} \in \mathsf{Y}^i(n)} \mathsf{y} \triangleright_i \mathfrak{Z}^i$.

For an outcome to be in the core, two conditions must be satisfied. First, it must be individually rational, in the sense that no agent would be better off by choosing to not participate. Second, there must not exist a set of contracts Y such that, for each type of agent associated with Y, the contracts may be partitioned so that, for each element of the partition, we can find an agent who prefers that element of the partition to his current assignment.

B.2 Reduction of the Multiunit Demand Economy to the Unit-Demand Economy

We now consider the unit-demand economy *induced* by the multiunit demand economy. The set of überroles $\mathcal{X}_i$ of Section 5 for each agent i is given by $\mathcal{X} \equiv (\mathcal{P}(\mathfrak{X}_i) \smallsetminus \varnothing) \cup \{o^i\}$, where o^i denotes the "outside option" of agent i of engaging in a set of roles $\varnothing \subseteq \mathfrak{X}_i$. The set of übercontracts X is given by $X \equiv \{x \subseteq \mathcal{X} : \chi, y \in x \Rightarrow \chi \cap y = \varnothing$ and $\exists \mathsf{Y} \subseteq \mathsf{X}$ such that $\cup_{\mathsf{y} \in \mathsf{Y}} \mathsf{y} = \cup_{\chi \in x} \chi\}$;

that is, the set of übercontracts is the set of subsets of überroles such that there exists a corresponding set of contracts representing the same underlying activities by agents.[29]

The preferences of agent i over subsets of roles induce preferences over überroles. We say that a strict ordering $\succ_i$ over $\mathcal{X}_i$ is *consistent* with the preferences of agent i if, for all $y, z \in \mathcal{X}_i$ such that $y \succ_i z$,

1. $y \neq o^i$ and $z \neq o^i$, then $y \trianglerighteq_i z$,
2. $y \neq o^i$ and $z = o^i$, then $y \trianglerighteq_i \varnothing$, and
3. $y = o^i$ and $z \neq o^i$, then $\varnothing \trianglerighteq_i z$.

We now define the transform $\mathbf{T}$, which transforms outcomes in the induced unit-demand economy into outcomes of the multiunit demand economy. For an outcome m of the unit-demand economy, we define, for each set of roles $\mathfrak{Y} \subseteq \mathfrak{X}_i$ of the multiunit demand economy,

$$\mathbf{T}^i_{\mathfrak{Y}}(m) \equiv \begin{cases} m^i_{\mathfrak{Y}} & \text{if } \mathfrak{Y} = y \text{ for some } y \in \mathcal{X}_i \\ m^i_{o^i} & \text{if } \mathfrak{Y} = \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

(Recall that roles are simply sets of überroles.) It is immediate that if m is an outcome of the induced unit-demand economy, then $\mathbf{T}(m)$ is an outcome of the multiunit demand economy. We now show that if an outcome m of the induced unit-demand economy satisfies the standard definition of the core in that setting, then $\mathbf{T}(m)$ satisfies Definition 9, i.e., is in the core of the multiunit demand economy.

Lemma 4. *Consider a core outcome m of the induced unit-demand economy, with preferences $\succ$ consistent with $\triangleright$. Then $\mathbf{T}(m)$ is in the core of the associated multiunit demand economy.*

[29] Note that, if we wish to allow an agent to perform duplicate versions of the same role, we can use multiple distinct roles to represent the same action by an agent. For instance, if roles denote one hour of work at a task, a set containing multiple of those roles would represent the agent working for multiple hours at that task.

Proof. Suppose that $\mathbf{T}(m)$ is not in the core of the multiunit demand economy. There are two cases.

1. There exists an agent i and set of roles $\mathfrak{Y} \subseteq \mathfrak{X}_i$ such that $\varnothing \rhd_i \mathfrak{Y}$ and $\mathbf{T}^i_{\mathfrak{Y}}(m) > 0$. But then m was not a core outcome of the induced unit-demand economy as $\succ$ is consistent with $\rhd$ and hence $o^i \succ_i \mathfrak{Y}$ and $m^i_{\mathfrak{Y}} > 0$.

2. There exists a set of contracts $\mathsf{Y} \subseteq \mathsf{X}$ such that, for each $i \in \mathsf{a}(\mathsf{Y})$, there exists a partition of Y_i into nonempty sets $\{\mathsf{Y}^i(n)\}_{n \in N(i)}$ and a set of roles $\mathfrak{Z}^i \subseteq \mathfrak{X}_i$ such that

   (a) $m^i_{\mathfrak{Z}^i} > 0$, and

   (b) for each $n \in N(i)$, $\cup_{\mathsf{y} \in \mathsf{Y}^i(n)} \mathsf{y} \rhd_i \mathfrak{Z}^i$.

   Consider the übercontract $x = \{\chi \in \mathcal{X} : \exists i \in I, \exists n \in N(i) \text{ such that } \chi = \cup_{\mathsf{y} \in Y^i(n)} \mathsf{y}_i\}$, and, for each $i \in \mathsf{a}(x)$, the überrole $z^i = \mathfrak{Z}^i$ in Condition 1 above. Then, for each $i \in \mathsf{a}(x)$,

   (a) $m^i_{z^i} > 0$ as $\mathbf{T}^i_{\mathfrak{Z}^i}(m) > 0$, and

   (b) for each $\chi \in x \cap \mathcal{X}_i$, $\chi \succ_i z^i$ as $\succ$ is consistent with $\rhd$ and $\chi = \cup_{\mathsf{y} \in Y^i(n)} \mathsf{y}_i$ for some $n \in N(i)$.

   Hence, m was not a core outcome of the induced unit-demand economy.

□

The following corollary then immediately follows from Lemma 4 and Theorem 2.

Corollary 1. *A core outcome of the multiunit demand economy exists.*