Appendix to "Strategy-Proofness in the Large"

Eduardo M. Azevedo and Eric Budish

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A Proofs

A.1 Proof of Theorem 1

We first define notation that will be used in the proof of Theorem 1. Given $\hat{\mu} \in \Delta T$, let $\Phi_i^n(t_i|\hat{\mu})$ denote the bundle $\Phi_i^n(t_i, t_{-i})$, where t_{-i} is an arbitrary vector of $n-1$ types such that $\text{emp}[t_i, t_{-i}] = \hat{\mu}$, if such t_{-i} exists.[1] If there is no such t_{-i}, which is the case for example if $\hat{\mu}(t_i) = 0$, then $\Phi_i^n(t_i|\hat{\mu})$ is defined as the random bundle placing equal weight on all outcomes in X_0. Note that bundles $\Phi_i^n(t_i|\hat{\mu})$ which do not correspond to any t_{-i} do not play any role in the results. They are defined only to simplify the notation in the proof below. Let $\Pr\{\hat{\mu}|t_i', \mu, n\}$ be the probability that the empirical distribution of (t_i', t_{-i}) is $\hat{\mu}$, given a fixed t_i' and that the vector t_{-i} of $n-1$ types is drawn i.i.d. according to μ. Throughout the proof we consider sums over infinite sets, but where only a finite number of the summands are nonzero. We adopt the convention that these are finite sums of only the positive terms.

Fix a prior $\mu \in \bar{\Delta}T$, market size n, and consider the utility a type t_i agent expects to obtain if she reports t_i'. This equals

$$u_{t_i}[\phi_i^n(t_i', \mu)] = \sum_{\hat{\mu} \in \Delta T} \Pr\{\hat{\mu}|t_i', \mu, n\} \cdot u_{t_i}[\Phi_i^n(t_i'|\hat{\mu})].$$

The interim gain from misreporting as type t_i' instead of type t_i equals

$$\begin{aligned} & u_{t_i}[\phi_i^n(t_i', \mu)] - u_{t_i}[\phi_i^n(t_i, \mu)] \\ = & \sum_{\hat{\mu} \in \Delta T} \Pr\{\hat{\mu}|t_i', \mu, n\} \cdot u_{t_i}[\Phi_i^n(t_i'|\hat{\mu})] - \sum_{\hat{\mu} \in \Delta T} \Pr\{\hat{\mu}|t_i, \mu, n\} \cdot u_{t_i}[\Phi_i^n(t_i|\hat{\mu})]. \end{aligned} \tag{A.1}$$

[1] Recall that anonymity implies that, if t_{-i} and t_{-i}' have the same empirical distribution, then $\Phi_i^n(t_i, t_{-i}) = \Phi_i^n(t_i, t_{-i}')$.

We can reorder the terms on the RHS of (A.1) as

$$\begin{aligned}&\underbrace{\sum_{\hat{\mu}\in\Delta T}\Pr\{\hat{\mu}|t_i,\mu,n\}\cdot\left(u_{t_i}[\Phi_i^n(t_i'|\hat{\mu})]-u_{t_i}[\Phi_i^n(t_i|\hat{\mu})]\right)}_{\text{Envy = Gain from reporting } t_i' \text{ holding fixed } \hat{\mu}} \\ &+\underbrace{\sum_{\hat{\mu}\in\Delta T}\left(\Pr\{\hat{\mu}|t_i',\mu,n\}-\Pr\{\hat{\mu}|t_i,\mu,n\}\right)\cdot u_{t_i}[\Phi_i^n(t_i'|\hat{\mu})]}_{\text{Gain from affecting } \hat{\mu}}.\end{aligned} \tag{A.2}$$

That is, the gain from misreporting can be decomposed into two terms. The first term is the expected gain, over all possible empirical distributions $\hat{\mu}$, of reporting t_i' instead of t_i, holding fixed the empirical distribution of types. This quantity equals how much type t_i players envy type t_i' players, in expectation. The second term is the sum, over all possible empirical distributions $\hat{\mu}$, of how much changing the report from t_i to t_i' increases the likelihood of $\hat{\mu}$, times the utility of receiving the bundle given to a type t_i' agent. That is, how much player i gains by manipulating the expected empirical distribution of reports $\hat{\mu}$. Our goal is to show that, if a mechanism is EF or EF-TB, then both of these terms are bounded above in large markets.

The proof is based on two lemmas. The first lemma bounds the effect that a single player can have on the probability distribution of the realized empirical distribution of types. This will allow us to bound the second term in expression (A.2).

Lemma A.1. *Define, given types t_i and t_i', distribution of types $\mu \in \Delta T$, and market size n, the function*

$$\Delta P(t_i,t_i',\mu,n)=\sum_{\hat{\mu}\in\Delta T}\left|\Pr\{\hat{\mu}|t_i',\mu,n\}-\Pr\{\hat{\mu}|t_i,\mu,n\}\right|. \tag{A.3}$$

Then, for any $\mu \in \bar{\Delta}T$, and $\epsilon > 0$, there exists a constant $C_{\Delta P} > 0$ such that, for any t_i, t_i' and n we have

$$\Delta P(t_i,t_i',\mu,n)\le C_{\Delta P}\cdot n^{-1/2+\epsilon}.$$

The second lemma will help us bound the first term in expression (A.2). Note that this term is always weakly negative for EF mechanisms, by definition, but that it can be positive for EF-TB mechanisms. The lemma provides a bound on the maximum amount of envy in an EF-TB mechanism, based on the minimum number of agents of a given type.

Lemma A.2. *Fix an EF-TB mechanism $\{(\Phi^n)_{\mathbb{N}}, T\}$. Define, given types t_i and t_i', empirical distribution of types $\hat{\mu} \in \Delta T$, and market size n, the function*

$$E(t_i, t_i', \hat{\mu}, n) = u_{t_i}[\Phi_i^n(t_i'|\hat{\mu})] - u_{t_i}[\Phi_i^n(t_i|\hat{\mu})],$$

which measures the envy of t_i for t_i'. Then, for any $\epsilon > 0$, there exists C_E such that, for all $t_i, t_i' \in T$, n, and $\hat{\mu} \in \bar{\Delta}T$ such that $\hat{\mu}$ corresponds to the empirical distribution of types for some vector in T^n, we have

$$E(t_i, t_i', \hat{\mu}, n) \leq C_E \cdot \min_{\tau \in T} \{\hat{\mu}(\tau) \cdot n\}^{-1/4+\epsilon}. \tag{A.4}$$

The proofs of Lemmas A.1 and A.2 are given below. We now use the two lemmas to prove Theorem 1

Proof of Theorem 1, Case 1: EF mechanisms. Applying the notation of Lemmas A.1 and A.2 to the terms in equation (A.2), and recalling that utility is bounded above by 1, we obtain the bound

$$\begin{aligned} u_{t_i}[\phi_i^n(t_i', \mu)] - u_{t_i}[\phi_i^n(t_i, \mu)] \quad \leq \quad & \sum_{\hat{\mu} \in \Delta T} \Pr\{\hat{\mu}|t_i, \mu, n\} \cdot E(t_i, t_i', \hat{\mu}, n) \\ & + \Delta P(t_i, t_i', \mu, n). \end{aligned} \tag{A.5}$$

If a mechanism is EF and $\hat{\mu}(t_i') > 0$, i.e., the empirical $\hat{\mu}$ has at least one report of t_i', then the first term in the RHS of inequality (A.5) is nonpositive. Taking any $\epsilon > 0$, and using Lemma A.1 to bound the ΔP term in the RHS of inequality (A.5) we have that there exists $C_{\Delta P} > 0$ such that

$$\begin{aligned} u_{t_i}[\phi_i^n(t_i', \mu)] - u_{t_i}[\phi_i^n(t_i, \mu)] \quad \leq \quad & \Pr\{\hat{\mu}(t_i') = 0|t_i, \mu, n\} \\ & + C_{\Delta P} \cdot n^{-1/2+\epsilon}. \end{aligned} \tag{A.6}$$

Since the probability that $\hat{\mu}(t_i') = 0$ goes to 0 exponentially with n, we have the desired result.

□

Proof of Theorem 1, Case 2: EF-TB mechanisms. We begin by bounding the envy term in inequality (A.5), which is weakly negative for EF mechanisms but can be strictly positive in

EF-TB mechanisms. We can, for any $\delta \geq 0$, decompose the envy term as

$$\sum_{\hat{\mu} \in \Delta T} \Pr\{\hat{\mu}|t_i, \mu, n\} \cdot E(t_i, t_i', \hat{\mu}, n) = \sum_{\hat{\mu}: \min_\tau \hat{\mu}(\tau) \geq \mu(\tau) - \delta} \Pr\{\hat{\mu}|t_i, \mu, n\} \cdot E(t_i, t_i', \hat{\mu}, n) + \sum_{\hat{\mu}: \min_\tau \hat{\mu}(\tau) < \mu(\tau) - \delta} \Pr\{\hat{\mu}|t_i, \mu, n\} \cdot E(t_i, t_i', \hat{\mu}, n). \tag{A.7}$$

By Lemma A.2, for any $\epsilon > 0$ there exists a constant C_E such that

$$\sum_{\hat{\mu}: \min_\tau \hat{\mu}(\tau) \geq \mu(\tau) - \delta} \Pr\{\hat{\mu}|t_i, \mu, n\} \cdot E(t_i, t_i', \hat{\mu}, n) \leq C_E \cdot \min_{\tau \in T} \{(\mu(\tau) - \delta)n\}^{-1/4+\epsilon}. \tag{A.8}$$

To bound the second term in the RHS of A.7, begin by noting that $\hat{\mu}(\tau) \cdot n$ equals the number of agents who draw type τ. This number is the outcome of $n-1$ i.i.d. draws of agents different than i, plus 1 if $t_i = \tau$. Using Hoeffding's inequality, for any τ, we can bound the probability that the realized value of $\hat{\mu}(\tau) \cdot n$ is much smaller than $\mu(\tau) \cdot n$. We have that, for any $\delta > 0$, there exists a constant $C_{\delta,\mu} > 0$ such that[2]

$$\Pr\{\hat{\mu}(\tau) \cdot n < (\mu(\tau) - \delta) \cdot n | t_i, \mu, n\} \leq C_{\delta,\mu} \cdot \exp\{-2\delta^2 n\}. \tag{A.9}$$

Take now $\delta = \min_{\tau \in T} \mu(\tau)/2$. Applying the bounds (A.8) and (A.9) to inequality (A.7), we have that

$$\sum_{\hat{\mu} \in \Delta T} \Pr\{\hat{\mu}|t_i, \mu, n\} \cdot E(t_i, t_i', \hat{\mu}, n) \leq C_E \cdot \min_{\tau \in T} \{(\mu(\tau) - \delta)n\}^{-1/4+\epsilon} + |T| \cdot C_{\delta,\mu} \cdot \exp\{-2\delta^2 n\}.$$

Multiplying n out of the first term in the RHS then yields

$$\sum_{\hat{\mu} \in \Delta T} \Pr\{\hat{\mu}|t_i, \mu, n\} \cdot E(t_i, t_i', \hat{\mu}, n) \leq C_E \cdot \min_{\tau \in T} \{\mu(\tau) - \delta\}^{-1/4+\epsilon} \cdot n^{-1/4+\epsilon} + |T| \cdot C_{\delta,\mu} \cdot \exp\{-2\delta^2 n\}.$$

[2] Hoeffding's inequality states that, given n i.i.d. binomial random variables with probability of success p, and $z > 0$, the probability of having fewer than $(p-z)n$ successes is bounded above by $\exp\{-2z^2 n\}$. Note that, in the bound below, t_i is fixed, while the $n-1$ coordinates of t_{-i} are drawn i.i.d. according to μ. For that reason, the Hoeffding bound must be modified to include a constant that depends on δ and μ, which we denote $C_{\delta,\mu}$. The reason why a constant suffices is that, conditional on δ and μ, the bound taking into account the $n-1$ draws converges to 0 at the same rate as the bound considering n draws.

Therefore, there exists a constant C' such that for all n, t_i', and t_i,

$$\sum_{\hat{\mu}\in\Delta T} \Pr\{\hat{\mu}|t_i,\mu,n\}\cdot E(t_i,t_i',\hat{\mu},n) \leq C'\cdot n^{-1/4+\epsilon}.$$

Return now to inequality (A.5). Using the bound we just derived and Lemma A.1, we have that there exists a constant $C_{\Delta P}$ such that

$$\begin{aligned} u_{t_i}[\phi_i^n(t_i',\mu)] - u_{t_i}[\phi_i^n(t_i,\mu)] \quad \leq \quad & C'\cdot n^{-1/4+\epsilon} \\ & + C_{\Delta P}\cdot n^{-1/2+\epsilon}. \end{aligned}$$

Therefore, there exists a constant C'' such that

$$u_{t_i}[\phi_i^n(t_i',\mu)] - u_{t_i}[\phi_i^n(t_i,\mu)] \leq C''\cdot n^{-1/4+\epsilon},$$

as desired.

□

A.1.1 Proof of the Lemmas

We now prove the lemmas. Throughout the proofs, we consider the case $\epsilon < 1/4$, which implies the results for $\epsilon \geq 1/4$.

Proof of Lemma A.1. To show that a single player cannot appreciably affect the distribution of $\hat{\mu}$, we start by calculating the effect of changing i's report on the probability of an individual value of $\hat{\mu}$ being drawn. Consider any $\hat{\mu}$ that is the empirical distribution of some vector of types with n agents.

Enumerate the elements of T as

$$T = \{\tau_1, \tau_2, \cdots \tau_{|T|}\}.$$

Since $\hat{\mu}$ follows a multinomial distribution, for any $t_i \in T$, the probability $\Pr\{\hat{\mu}|t_i,\mu,n\}$ equals

$$\binom{n-1}{n\hat{\mu}(\tau_1),\cdots,n\hat{\mu}(t_i)-1,\cdots,n\hat{\mu}(\tau_{|T|})}\cdot \mu(\tau_1)^{n\hat{\mu}(\tau_1)}\cdots\mu(t_i)^{n\hat{\mu}(t_i)-1}\cdots\mu(\tau_{|T|})^{n\hat{\mu}(\tau_{|T|})},$$

where the term in parentheses is a multinomial coefficient. Note that the $n\hat{\mu}(\tau)$ terms in this expression are integers, since this is the number of agents with a given type in a realization

$\hat{\mu}$ of the distribution of types. Moreover, t_i only enters the formula in one factorial term in the denominator, and a power term in the numerator. With this observation, we have that

$$\Pr\{\hat{\mu}|t_i', \mu, n\}/\Pr\{\hat{\mu}|t_i, \mu, n\} = \frac{\hat{\mu}(t_i')}{\mu(t_i')}/\frac{\hat{\mu}(t_i)}{\mu(t_i)}. \tag{A.10}$$

For the rest of the proof, we will consider separately values of $\hat{\mu}$ which are close to μ, and those that are very different from μ. We will show that player i can only have a small effect on the probability of the former, while the latter occur with very small probability.

We derive bounds as functions of a variable δ. Initially, we derive bounds valid for any $\delta > 0$, and, later in the proof, we consider the case where δ is a particular function of n. Define, for any $\delta > 0$, the set M_δ of empirical distributions $\hat{\mu}$ that are sufficiently close to the true distribution μ as

$$M_\delta = \{\hat{\mu} \in \Delta T : |\hat{\mu}(t_i) - \mu(t_i)| < \delta \text{ and } |\hat{\mu}(t_i') - \mu(t_i')| < \delta\}.$$

Note that, when $\hat{\mu}(t_i) = \mu(t_i)$ and $\hat{\mu}(t_i') = \mu(t_i')$, the ratio on the right of equation (A.10) equals 1 and is continuously differentiable in $\hat{\mu}(t_i)$ and $\hat{\mu}(t_i')$. Consequently, there exists a constant $C > 0$, and $\bar{\delta} > 0$ such that, for all $\delta \leq \bar{\delta}$, if $\hat{\mu} \in M_\delta$ then

$$|\frac{\hat{\mu}(t_i')}{\mu(t_i')}/\frac{\hat{\mu}(t_i)}{\mu(t_i)} - 1| < C\delta. \tag{A.11}$$

Moreover, we can bound the probability that the empirical distribution of types $\hat{\mu}$ is not in

$M_{\delta+\frac{1}{n}}$. By Hoeffding's inequality,[3] for any $\delta > 0$ and n,

$$\begin{aligned}\Pr\{\hat{\mu} \notin M_{\delta+\frac{1}{n}} | t_i, \mu, n\} &\leq 4 \cdot \exp(-2(n-1)\delta^2) \\ \Pr\{\hat{\mu} \notin M_{\delta+\frac{1}{n}} | t_i', \mu, n\} &\leq 4 \cdot \exp(-2(n-1)\delta^2).\end{aligned} \tag{A.12}$$

We are now ready to bound ΔP. We can decompose the sum in equation (A.3) into the terms where $\hat{\mu}$ is within or outside $M_{\delta+\frac{1}{n}}$. We then have

$$\begin{aligned}\Delta P &= \sum_{\hat{\mu} \in M_{\delta+\frac{1}{n}}} |\Pr\{\hat{\mu}|t_i', \mu, n\} - \Pr\{\hat{\mu}|t_i, \mu, n\}| \\ &\quad + \sum_{\hat{\mu} \notin M_{\delta+\frac{1}{n}}} |\Pr\{\hat{\mu}|t_i', \mu, n\} - \Pr\{\hat{\mu}|t_i, \mu, n\}|.\end{aligned}$$

Rearranging the first term, and using the triangle inequality in the second term we have

$$\begin{aligned}\Delta P &\leq \sum_{\hat{\mu} \in M_{\delta+\frac{1}{n}}} |\Pr\{\hat{\mu}|t_i', \mu, n\} / \Pr\{\hat{\mu}|t_i, \mu, n\} - 1| \cdot \Pr\{\hat{\mu}|t_i, \mu, n\} \\ &\quad + \sum_{\hat{\mu} \notin M_{\delta+\frac{1}{n}}} (\Pr\{\hat{\mu}|t_i', \mu, n\} + \Pr\{\hat{\mu}|t_i, \mu, n\}).\end{aligned}$$

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[3] Hoeffding's inequality yields

$$\Pr\{|\hat{\mu}(t_i) - \frac{n-1}{n}\mu(t_i) - \frac{1}{n}| > \delta | t_i, \mu, n\} < 2\exp\{-2(n-1)\delta^2\}.$$

Moreover,

$$\begin{aligned}|\hat{\mu}(t_i) - \mu(t_i)| &= |\hat{\mu}(t_i) - \frac{n-1}{n}\mu(t_i) - \frac{1}{n} + \frac{1}{n}(1 - \mu(t_i))| \\ &\leq |\hat{\mu}(t_i) - \frac{n-1}{n}\mu(t_i) - \frac{1}{n}| + \frac{1}{n}|1 - \mu(t_i)|.\end{aligned}$$

Hence,

$$\Pr\{|\hat{\mu}(t_i) - \mu(t_i)| > \delta + \frac{1}{n} | t_i, \mu, n\} < 2\exp\{-2(n-1)\delta^2\}.$$

By a similar argument,

$$\Pr\{|\hat{\mu}(t_i') - \mu(t_i')| > \delta + \frac{1}{n} | t_i, \mu, n\} < 2\exp\{-2(n-1)\delta^2\}.$$

Adding these two bounds implies the bound (A.12) when player i plays t_i, and the case where player i plays t_i' is analogous.

If we substitute equation (A.10) in the first term we obtain

$$\begin{aligned}\Delta P &\leq \sum_{\hat{\mu}\in M_{\delta+\frac{1}{n}}} |\frac{\hat{\mu}(t_i')}{\mu(t_i')}/\frac{\hat{\mu}(t_i)}{\mu(t_i)} - 1| \cdot \Pr\{\hat{\mu}|t_i,\mu,n\} \\ &\quad + \sum_{\hat{\mu}\notin M_{\delta+\frac{1}{n}}} (\Pr\{\hat{\mu}|t_i',\mu,n\} + \Pr\{\hat{\mu}|t_i,\mu,n\}).\end{aligned}$$

We can bound the first sum using the fact that the ratio being summed is small for $\hat{\mu} \in M_{\delta+\frac{1}{n}}$, and bound the second sum since the total probability that $\hat{\mu} \notin M_{\delta+\frac{1}{n}}$ is small. Formally, using equations (A.11) and (A.12) we have that, for all n and δ with $\delta + \frac{1}{n} \leq \bar{\delta}$,

$$\Delta P \leq C(\delta + \frac{1}{n}) + 8 \cdot \exp(-2(n-1)\delta^2).$$

To complete the proof we will substitute δ by an appropriate function of n. Note that the first term is increasing in δ, while the second term is decreasing in δ. In particular, for the second term to converge to 0, asymptotically δ has to be greater than $n^{-1/2}$. If we take $\delta = n^{-1/2+\epsilon}$, we obtain the bound

$$\Delta P \leq C(n^{-1/2+\epsilon} + n^{-1}) + 8 \cdot \exp(-2n^{2\epsilon}\frac{n-1}{n}), \tag{A.13}$$

for all n large enough such that $\delta + \frac{1}{n} = n^{-1/2+\epsilon} + n^{-1} \leq \bar{\delta}$. Therefore, we can take a constant C' such that

$$\Delta P \leq C' \cdot (n^{-1/2+\epsilon} + \exp(-2n^{2\epsilon}\frac{n-1}{n})) \tag{A.14}$$

for all n.

Asymptotically, the first term in the RHS of (A.14) dominates the second term.[4] Therefore, we can find a constant $C_{\Delta P}$ such that

$$\Delta P \leq C_{\Delta P} \cdot n^{-1/2+\epsilon},$$

completing the proof.

□

We now prove Lemma A.2. The result would follow immediately if we restricted attention to mechanisms that are EF. The difficulty in establishing the result is that mechanisms that

[4] To see this, note that the logarithm of $n^{-1/2+\epsilon}$ is $-(1/2+\epsilon)\log n$, while the logarithm of $\exp(-2n^{2\epsilon}\frac{n-1}{n})$ equals $-2n^{2\epsilon}\frac{n-1}{n}$. Since $n^{2\epsilon}\frac{n-1}{n}$ is asymptotically much larger than $\log n$, we have that the second term in equation (A.13) is asymptotically much smaller than the first.

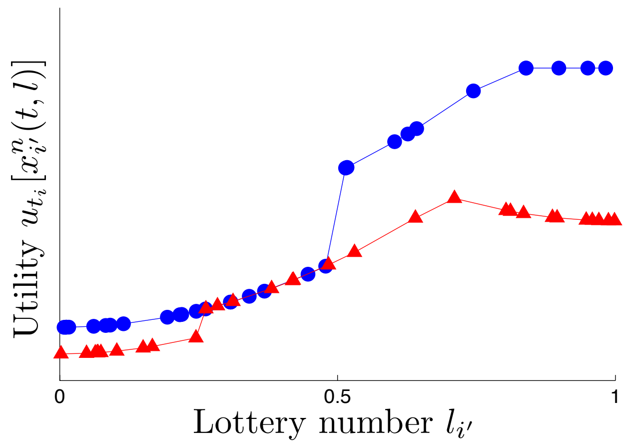

Figure A.1: A scatter plot of the lottery numbers $l_{i'}$ of different agents i' on the horizontal axis, and the utility $u_{t_i}[x^n_{i'}(t,l)]$ of type t_i agents from the bundles i' receives in the vertical axis. Balls represent agents with $t_{i'} = t_i$, and triangles agents with $t_{i'} = t_j$. The values are consistent with EF-TB, as the utilities of type t_i agents are always above the utilities from bundles of any agent with lower lottery number.

are EF-TB but not EF can have large amounts of envy ex-post, i.e., $u_{t_i}[\Phi^n_j(t)] - u_{t_i}[\Phi^n_i(t)]$ can be large. To see why this can be the case, fix two players i and j and consider Figure A.1. The figure plots, for several players i' whose types are either $t_{i'} = t_i$ or $t_{i'} = t_j$, lottery numbers $l_{i'}$ in the horizontal axis and the utility of a type t_i for the bundle i' receives in the vertical axis. Players with $t_{i'} = t_i$ are plotted as balls, and players with $t_{i'} = t_j$ as triangles. Note that the figure is consistent with EF-TB. In particular, if $l_j \leq l_i$, then player i prefers his own bundle to player j's bundle. However, if player j received a higher lottery number, $l_j > l_i$, it is perfectly consistent with EF-TB that player i prefers player j's bundle. That is, a player corresponding to a ball may envy a player corresponding to a triangle in the picture, as long as the triangle player has a higher lottery number. In fact, player i can envy player j by a large amount, so EF-TB mechanisms can have a lot of envy ex-post.

Figure A.1 also suggests a way to prove the lemma, despite this difficulty. The proof exploits two basic insights. First, note that the curve formed by the balls – the utility player i derives from the bundles assigned to the type t_i players – is always above the curve formed by the triangles – the utility player i derives from the bundles assigned to the type t_j players. Hence, for type t_i agents to, on average, have a large amount of ex-post envy of type t_j agents, the lottery outcome must be very uneven, favoring type t_j players over type t_i players. We can bound this average ex-post envy as a function of how well distributed lottery numbers are (see Claim A.1). Second, due to symmetry, how much player i envies player j ex-ante (i.e., before the lottery) equals how much player i prefers the bundles received by type t_j

players over the bundles received by type t_i players, averaging over all type t_i and t_j players, and all possible lottery draws. Since lottery draws are likely to be very evenly distributed in a large market, it follows that player i's envy with respect to player j, before the lottery draw, is small (see Claim A.2). We now formalize these ideas.

Proof of Lemma A.2. The proof of the lemma has three steps. The first step bounds how much players of a given type envy players of another type, on average, conditional on a vector of reports t and lottery draw l, as a function of how evenly distributed the lottery numbers are. The second step bounds envy between two players, conditional on a vector of reports t, but before the lottery is drawn. Finally, the third step uses these bounds to prove the result.

Step 1. Bounding average envy after a lottery draw.

We begin by defining a measure of how evenly distributed a vector of lottery numbers is. Fix a market size n, vector of types $t \in T^n$, vector of lottery draws l and players i and j. Partition the set of players in groups according to where their lottery number falls among K uniformly-spaced intervals $L_1 = [0, 1/K)$, $L_2 = [1/K, 2/K)$, $\cdots, L_K = [(K-1)/K, 1]$. Denote the set of all type $t_{i'}$ players by

$$I(i'|t) = \{i'' : t_{i''} = t_{i'}\},$$

and denote the set of type $t_{i'}$ players with lottery numbers in L_k by

$$I_k(i'|t, l) = \{i'' \in I(i'|t) : l_{i''} \in L_k\}.$$

When there is no risk of confusion, these sets will be denoted by $I(i')$ and $I_k(i')$, respectively. The number of elements in a set of players $I(i')$ is denoted by $|I(i')|$.

Given the lottery draw l, we choose the number of partitions $K(l, t, i, j)$ such that the type t_i and type t_j players' lottery numbers are not too unevenly distributed over the L_k sets. Specifically, let $K(l, t, i, j)$ be the largest integer K such that, for $i' = i, j$, and $k = 1, \cdots, K$, we have

$$\left|\frac{|I_k(i'|t, l)|}{|I(i'|t)|} - \frac{1}{K}\right| < \frac{1}{K^2}. \tag{A.15}$$

Such an integer necessarily exists, as $K = 1$ satisfies this condition. Intuitively, the larger is $K(l, t, i, j)$, the more evenly distributed the lottery numbers l are. When there is no risk of confusion, we write $K(l)$ or K for $K(l, t, i, j)$.

The following claim bounds the average envy of type t_i players towards type t_j players, after a lottery draw, as a function of $K(l, t, i, j)$.

Claim A.1. Fix a market size n, vector of types $t \in T^n$, lottery draws $l \in [0,1]^n$, and players i and j. Then the average envy of type t_i players towards type t_j players is bounded by

$$\sum_{j' \in I(j)} \frac{u_{t_i}[x^n_{j'}(t,l)]}{|I(j|t)|} - \sum_{i' \in I(i)} \frac{u_{t_i}[x^n_{i'}(t,l)]}{|I(i|t)|} \leq \frac{3}{K(l,t,i,j)}. \tag{A.16}$$

Proof. Denote the minimum utility received by a player with type t_i and lottery number in L_k as

$$v_k(l) = \min\{u_{t_i}[x^n_{i'}(t,l)] : i' \in I_k(i)\}.$$

Define $v_{K(l)+1}(l) = 1$. Although $v_k(l)$ and $K(l)$ depend on l, we will omit this dependence when there is no risk of confusion. Note that, by the EF-TB condition, for all $j' \in I_k(j)$,

$$u_{t_i}[x^n_{j'}(t,l)] \leq v_{k+1}. \tag{A.17}$$

Moreover, for all $i' \in I_{k+1}(i)$,

$$v_{k+1} \leq u_{t_i}[x^n_{i'}(t,l)]. \tag{A.18}$$

We now bound the average utility a type t_i agent derives from the bundles received by all players with type t_j as follows.

$$\begin{aligned} & \sum_{j' \in I(j)} \frac{u_{t_i}[x^n_{j'}(t,l)]}{|I(j)|} && \text{(A.19)} \\ = & \sum_{k=1}^{K} \sum_{j' \in I_k(j)} \frac{|I_k(j)|}{|I(j)|} \cdot \frac{u_{t_i}[x^n_{j'}(t,l)]}{|I_k(j)|} \\ \leq & \sum_{k=1}^{K} \frac{|I_k(j)|}{|I(j)|} \cdot v_{k+1}. \end{aligned}$$

The second line follows from breaking the sum over the K sets $I_k(j)$, and the third line follows from inequality (A.17). We now use the fact that K was chosen such that both $|I_k(i)|/|I(i)|$ and $|I_k(j)|/|I(j)|$ are approximately equal to $1/K$. Using condition (A.15) we can bound the expression above as

$$\begin{aligned}\sum_{k=1}^{K}\frac{|I_k(j)|}{|I(j)|}\cdot v_{k+1} &= \sum_{k=2}^{K}\frac{|I_k(i)|}{|I(i)|}\cdot v_k+\sum_{k=2}^{K}[\frac{|I_{k-1}(j)|}{|I(j)|}-\frac{|I_k(i)|}{|I(i)|}]\cdot v_k+\frac{|I_K(j)|}{|I(j)|}\cdot v_{K+1}\\ &\leq \sum_{k=2}^{K}\frac{|I_k(i)|}{|I(i)|}\cdot v_k+(K-1)\frac{2}{K^2}+(\frac{1}{K}+\frac{1}{K^2})\\ &\leq \sum_{k=2}^{K}\frac{|I_k(i)|}{|I(i)|}\cdot v_k+\frac{3}{K}.\end{aligned}$$

The equation in the first line follows from rearranging the sum. The second line follows from $v_k \leq 1$, and from the fact that the fractions $I_k(i)/I(i)$ and $I_k(j)/I(j)$ are in the interval $[\frac{1}{K}-\frac{1}{K^2},\frac{1}{K}+\frac{1}{K^2}]$ as per inequality (A.15). The inequality in the third line follows from summing the second and third terms of the RHS of the second line.

We now bound the RHS of this expression using the fact that type t_i agents in the interval $I_k(i)$ receive utility of at least v_k. Using inequality (A.18) we have

$$\begin{aligned}&\sum_{k=2}^{K}\frac{|I_k(i)|}{|I(i)|}\cdot v_k+\frac{3}{K}\\ \leq\ &\sum_{k=2}^{K}\sum_{i'\in I_k(i)}\frac{|I_k(i)|}{|I(i)|}\cdot\frac{u_{t_i}[x^n_{i'}(t,l)]}{|I_k(i)|}+\frac{3}{K}\\ \leq\ &\sum_{k=1}^{K}\sum_{i'\in I_k(i)}\frac{|I_k(i)|}{|I(i)|}\cdot\frac{u_{t_i}[x^n_{i'}(t,l)]}{|I_k(i)|}+\frac{3}{K}.\end{aligned}$$

The first inequality follows from v_k being lower than the utility of any player in $I_k(i)$, and the second inequality follows because the latter sum equals the first plus the $k=1$ term. Since we started from inequality (A.19), the bound (A.16) follows, completing the proof.

□

Step 2: Bounding envy before the lottery draw.

We now bound the envy between two players i and j given a profile of types t, before the lottery is drawn.

Claim A.2. Given $\epsilon > 0$, there exists a constant $C_E > 0$ such that, for any $t \in T^n$ and $i, j \leq n$, player i's envy with respect to player j is bounded by

$$u_{t_i}[\Phi^n_j(t)] - u_{t_i}[\Phi^n_i(t)] \leq C_E \cdot \min_{i'=i,j}\{|I(i|t)|\}^{-1/4+\epsilon} \tag{A.20}$$

Proof. Given a vector of types t and a player i', using anonymity, we can write the expected bundle $\Phi^n_{i'}(t)$ received by player i' as the expected bundle received by all players with the same type, over all realizations of l:

$$\Phi^n_{i'}(t) = \int_{l\in[0,1]^n} \sum_{i''\in I(i')} \frac{x^n_{i''}(t,l)}{|I(i')|}\, dl. \tag{A.21}$$

Hence, player i's envy of player j can be written as:

$$u_{t_i}[\Phi^n_j(t)] - u_{t_i}[\Phi^n_i(t)] = \int_{l\in[0,1]^n} \sum_{j'\in I(j)} \frac{u_{t_i}[x^n_{j'}(t,l)]}{|I(j|t)|} - \sum_{i'\in I(i)} \frac{u_{t_i}[x^n_{i'}(t,l)]}{|I(i|t)|}\, dl.$$

Claim A.1 then implies that envy is bounded by

$$u_{t_i}[\Phi^n_j(t)] - u_{t_i}[\Phi^n_i(t)] \le \int_{l\in[0,1]^n} \frac{3}{K(l,t,i,j)}\, dl. \tag{A.22}$$

We need to show that, on average over all lottery realizations, $K(l)$ is large enough such that the integral above is small. Given a lottery draw l denote by $\hat{F}_{i'}(x|l)$ the fraction of agents in $I(i')$ with lottery number no greater than x. Formally,

$$\hat{F}_{i'}(x|l) = |\{i'' \in I(i') : l_{i''} \le x\}|/|I(i')|.$$

That is, $\hat{F}_{i'}$ is the empirical distribution function of the lottery draws of type $t_{i'}$ agents. Since the lottery numbers are i.i.d., we know that the $\hat{F}_{i'}(x|l)$ functions are very likely to be close to the actual distribution of lottery draws $F(x) = x$. By the Dvoretzky–Kiefer–Wolfowitz inequality, for any $\delta > 0$,

$$\Pr\{\sup_x |\hat{F}_{i'}(x|l) - x| > \delta\} \le 2\exp(-2|I(i')|\delta^2). \tag{A.23}$$

Fixing a partition size K, the conditions in (A.15) for the number of agents in each interval to be close to $1/K$ can be written as

$$|[\hat{F}_{i'}(\frac{k}{K}|l) - \hat{F}_{i'}(\frac{k-1}{K}|l)] - \frac{1}{K}| \le \frac{1}{K^2},$$

for $k = 1,\dots,K$ and $i' = i, j$. Applying the inequality (A.23), using $\delta = 1/2K^2$, we have that the probability that each such condition is violated is bounded by

$$\Pr\{|\frac{|I_k(i')|}{|I(i')|} - \frac{1}{K}| > \frac{1}{K^2}\} \leq 2 \cdot \exp(-|I(i')|/2K^4).$$

Consider now an arbitrary integer $\bar{K} > 0$. Note that the probability that $K(l) \geq \bar{K}$ is at least as large as the probability that $K = \bar{K}$ satisfies all of the conditions (A.15), since $K(l)$ by construction is the largest integer that satisfies these conditions. Therefore,

$$\begin{aligned}\Pr\{K(l) < \bar{K}\} &\leq 2\bar{K}[\exp(-|I(i)|/2\bar{K}^4) + \exp(-|I(j)|/2\bar{K}^4)] \\ &\leq 4\bar{K}\exp(-\min_{i'=i,j}\{|I(i')|\}/2\bar{K}^4).\end{aligned}$$

Using this, we can bound the integral in the right side of equation (A.22). Note that the integrand $3/K(l)$ is decreasing in $K(l)$, and attains its maximum value of 3 when $K(l) = 1$. Therefore, the integral in equation (A.22) can be bounded by

$$\begin{aligned}\int_{l\in[0,1]^n} \frac{3}{K(l,t,i,j)}\, dl &\leq \frac{3}{\bar{K}} + 3\Pr\{K(l) < \bar{K}\} \\ &\leq \frac{3}{\bar{K}} + 12\bar{K}\exp(-\min_{i'=i,j}\{|I(i')|\}/2\bar{K}^4),\end{aligned}$$

Note that the first term on the RHS is decreasing in $\bar{K}$, while the second term is increasing in $\bar{K}$. Taking $\bar{K} = \lfloor \min_{i'=i,j} |I(i')|^{1/4-\epsilon} \rfloor$, we have that this last expression is bounded by

$$\begin{aligned}&3/\min_{i'=i,j}\lfloor\{|I(i')|\}^{1/4-\epsilon}\rfloor \\ &+12\min_{i'=i,j}\{|I(i')|\}^{1/4-\epsilon}\exp\{-\min_{i'=i,j}\{|I(i')|\}^{4\epsilon}/2\}.\end{aligned}$$

Note that, as $\min_{i'=i,j}\{|I(i')|\}$ grows, the second term is asymptotically negligible compared to the first term.[5] Therefore, there exists a constant C_E such that equation (A.20) holds, proving the claim.

[5]This can be shown formally by taking logs of both terms. The log of the first term equals approximately

$$\log 3 - (\frac{1}{4} - \epsilon)\log\min_{i'=i,j}\{|I(i')|\},$$

while the log of the second term equals

$$\log 12 + (\frac{1}{4} - \epsilon)\log\min_{i'=i,j}\{|I(i')|\} - \min_{i'=i,j}\{|I(i')|\}^{4\epsilon}/2.$$

As $\min_{i'=i,j}\{|I(i')|\}$ grows, the difference between the second term and the first term goes to $-\infty$, because $\min_{i'=i,j}\{|I(i')|\}^{4\epsilon}$ grows much more quickly than $\log\min_{i'=i,j}\{|I(i')|\}$.

□

Step 3: Completing the proof.

The lemma now follows from Claim A.2. Take $\epsilon > 0$, and consider a constant C_E as in the statement of Claim A.2. Consider t_i, t_i', $\hat{\mu}$, and n as in the statement of the lemma. Recall that, since $\hat{\mu} \in \bar{\Delta}T$, we have $\hat{\mu}(\tau) > 0$ for all $\tau \in T$. Additionally, since $\hat{\mu}$ equals the empirical distribution of some vector of types, there exists t_{-i} and j such that $\hat{\mu} = \mathrm{emp}[t]$ and $t_j = t_i'$. Therefore, we have

$$\begin{aligned} E(t_i, t_i', \hat{\mu}, n) &= u_{t_i}[\Phi_i^n(t_i'|\hat{\mu})] - u_{t_i}[\Phi_i^n(t_i|\hat{\mu})] \\ &= u_{t_i}[\Phi_j^n(t)] - u_{t_i}[\Phi_i^n(t)] \\ &\leq C_E \cdot \min_{i'=i,j}\{|I(i|t)|\}^{-1/4+\epsilon} \\ &\leq C_E \cdot \min_{\tau \in T}\{\hat{\mu}(\tau) \cdot n\}^{-1/4+\epsilon}. \end{aligned}$$

The first equation is the definition of $E(t_i, t_i', \hat{\mu}, n)$. The equation in the second line follows from the way we defined t. The inequality in the third line follows from Claim A.2. The final inequality follows because $\min_{i'=i,j}\{|I(i|t)|\}$ is weakly greater than $\min_{\tau \in T}\{\hat{\mu}(\tau) \cdot n\}$.

□

A.1.2 Infinite Set of Bundles

We close this Section by highlighting that the assumption of a finite set of bundles X_0 is not necessary for Theorem 1.

Remark 1. For the proof of Theorem 1 and Lemmas A.1 and A.2, we do not have to assume X_0 finite. The proofs follow verbatim with the following assumptions. X_0 is a measurable subset of Euclidean space. Agents' utility functions over X_0 are measurable and have range $[-\infty, 1]$. The utility of reporting truthfully is at least 0. That is, for all n and $t \in T^n$,

$$u_{t_i}[\Phi_i^n(t)] \geq 0.$$

The theorem holds with otherwise arbitrary X_0 satisfying these assumptions. The added generality is important for classifying the Walrasian mechanism in Appendix C.1.4.

A.2 Proof of Theorem 2

Because $(F^n)_{n\in\mathbb{N}}$ is limit Bayes-Nash implementable, there exists a mechanism $((\Phi^n)_{n\in\mathbb{N}}, A)$ with a limit Bayes Nash equilibrium σ^* such that

$$F^n(\omega) = \Phi^n(\sigma^*(\omega))$$

for all n and vectors of n types ω in Ω_n^*. Define the direct mechanism $((\Psi^n)_{n\in\mathbb{N}}, T)$ by

$$\Psi^n(t) = \Phi^n(\sigma^*((t_1, \mathrm{emp}[t]), \ldots, (t_n, \mathrm{emp}[t]))).$$

Denote by $\psi^n(t_i, \mu)$ the bundle a participant who reports t_i expects to receive from Ψ^n if the other participants report i.i.d. according to μ.

Part 1: $((\Psi^n)_{n\in\mathbb{N}}, T)$ **approximately implements** $(F^n)_{n\in\mathbb{N}}$**.**

We must prove that, given t_i in T, μ in $\bar{\Delta}T$, and $\epsilon > 0$, there exists n_0 such that, for all $n \geq n_0$

$$\|f^n(t_i, \mu) - \psi^n(t_i, \mu)\| < \epsilon. \tag{A.24}$$

By the definition of $f^n(t_i, \mu)$ we have

$$f^n(t_i, \mu) = \sum_{t_{-i}\in T^{n-1}} \Pr\{t_{-i} | t_{-i} \sim iid(\mu)\} \cdot F_i^n((t_1, \mu), \ldots, (t_n, \mu)).$$

Likewise, by the definition of $\psi^n(t_i, \mu)$ we have

$$\begin{aligned}\psi^n(t_i, \mu) &= \sum_{t_{-i}\in T^{n-1}} \Pr\{t_{-i} | t_{-i} \sim iid(\mu)\} \cdot \Phi^n(\sigma^*((t_1, \mathrm{emp}[t]), \ldots, (t_n, \mathrm{emp}[t])))\\ &= \sum_{t_{-i}\in T^{n-1}} \Pr\{t_{-i} | t_{-i} \sim iid(\mu)\} \cdot F^n((t_1, \mathrm{emp}[t]), \ldots, (t_n, \mathrm{emp}[t])).\end{aligned}$$

Therefore, by the triangle inequality,

$$\|f^n(t_i, \mu) - \psi^n(t_i, \mu)\| \leq \sum_{t_{-i}\in T^{n-1}} \Pr\{t_{-i} | t_{-i} \sim iid(\mu)\} \cdot \Delta(t_{-i}), \tag{A.25}$$

where

$$\Delta(t_{-i}) = \|F_i^n((t_1, \mu), \ldots, (t_n, \mu)) - F_i^n((t_1, \mathrm{emp}[t]), \ldots, (t_n, \mathrm{emp}[t]))\|.$$

Moreover, because the social choice function $(F^n)_{n\in\mathbb{N}}$ is continuous, there exists a neighborhood $\mathcal{N}$ of μ and n_0 in $\mathbb{N}$ such that, for any t_{-i} with $\text{emp}[t] \in \mathcal{N}$ and $n \geq n_0$,

$$\Delta(t_{-i}) < \epsilon/2.$$

By the law of large numbers, we can take n_0 to be large enough so that the probability that $\text{emp}[t] \notin \mathcal{N}$ is lower than $\epsilon/2$.

We can decompose the difference in inequality (A.25) as

$$\begin{aligned} \|f^n(t_i,\mu) - \psi^n(t_i,\mu)\| \leq & \sum_{t_{-i}:\text{emp}[t]\in\mathcal{N}} \Pr\{t_{-i}|t_{-i} \sim iid(\mu)\} \cdot \Delta(t_{-i}) \\ & + \sum_{t_{-i}:\text{emp}[t]\notin\mathcal{N}} \Pr\{t_{-i}|t_{-i} \sim iid(\mu)\} \cdot \Delta(t_{-i}). \end{aligned}$$

Each of the terms on the right hand side is bounded above by $\epsilon/2$, which establishes inequality (A.24).

Part 2: $((\Phi^n)_{n\in\mathbb{N}}, T)$ is SP-L.

We must show that, for any t_i and t'_i in T, μ in $\bar{\Delta}T$, and $\epsilon > 0$, there exists n_0 such that, for all $n \geq n_0$,

$$u_{t_i}[\psi^n(t'_i,\mu)] - u_{t_i}[\psi^n(t_i,\mu)] \leq \epsilon. \tag{A.26}$$

From the triangle inequality we have that

$$\begin{aligned} u_{t_i}[\psi^n(t'_i,\mu)] - u_{t_i}[\psi^n(t_i,\mu)] \leq & u_{t_i}[f^n(t'_i,\mu)] - u_{t_i}[f^n(t_i,\mu)] \\ & + \|f^n(t'_i,\mu) - \psi^n(t'_i,\mu)\| \\ & + \|f^n(t_i,\mu) - \psi^n(t_i,\mu)\|. \end{aligned}$$

By the definition of f^n and the fact that σ^* is a limit Bayes-Nash equilibrium, there exists n_0 such that, for $n \geq n_0$, the first term in the right-hand side is bounded above by $\epsilon/3$. Moreover, by step 1 of this proof, we can take n_0 such that the second and third terms are bounded above by $\epsilon/3$. This implies inequality (A.26).

A.3 Proof of Corollary 1

In this section, we denote the Boston mechanism by $((\Phi)_{n\in\mathbb{N}}, S)$. The corollary uses some facts about limit equilibria of the boston mechanism given a common identically independently distributed prior over payoff types. Let $\Sigma^*(\mu)$ denote the set of limit equilibria of the Boston mechanism given a prior μ in ΔT . Formally, denote by Σ^{**} be the set of limit

Bayes-Nash equilibria of the Boston mechanism in the type space Ω^*. Then we define

$$\Sigma^*(\mu) = \{\rho \in \mathbb{R}_+^{T\times S} : \exists \sigma^* \in \Sigma^{**} \text{ such that } \rho(s,t_i) = \sigma^*(s,(t_i,\mu)) \text{ for all } s \in S, t_i \in T\}.$$

That is, each element ρ of $\Sigma^*(\mu)$ specifies the probability $\rho(s,t_i)$ with which type t_i agents play action s in a limit equilibrium of the game with a common identically idenpendently distributed prior μ over payoff types. In other words, ρ is an equilibrium strategy profile of the Boston mechanism with set of types T and a common iid prior μ. Let $P^*(\mu)$ be the set of vectors of probability of acceptance to each school in equilibrium. We then have the following result:

Proposition 1. *The correspondence $\Sigma^*(\mu)$ is non- empty, convex-valued and continuous in $\bar{\Delta}T$. The correspondence $P^*(\mu)$ is non-empty, single-valued, and continuous in $\bar{\Delta}T$.*

The Proposition shows that, given a prior μ, the Boston mechanism may have multiple equilibria. Nevertheless, the probability of acceptance to each school is the same in any equilibrium. The intuition is that lowering the probability of acceptance to a school weakly reduces the set of students who want to point to it, and weakly increases the set of students who want to point to other schools. Therefore, an argument similar to uniqueness arguments in competitive markets with gross substitutes shows that equilibrium probabilities of acceptance are unique. Moreover, equilibrium delivers well-behaved outcomes because probabilities of acceptance vary continuously.

Before proving the Proposition, we use it to establish Corollary 1.

Proposition 1 implies that Σ^* is non-empty, lower hemi-continuous, and convex-valued. The Michael Selection Theorem implies that Σ^* has a continuous selection. Thus, there exists a limit Bayes Nash equilibrium σ^* of the Boston mechanism defined over the type space Ω^*, and moreover this equilibrium $\sigma^*(t_i,\mu)$ varies continuously with μ in $\bar{\Delta}T$. Because outcomes of the Boston mechanism vary continuously with the empirical distribution of types, the social choice function $(F^n)_{n\in\mathbb{N}}$ defined by

$$F^n(\omega) = \Phi^n(\sigma^*(\omega))$$

is continuous and limit Bayes-Nash implementable. Corollary 1 then follows from Theorem 2.

A.3.1 Proof of Proposition 1

The Boston mechanism has a limit

$$\phi^{\infty}(s, m) = \min\{\frac{q_s}{m_s}, 1\}.$$

Therefore, a strategy profile ρ^* is in $\Sigma^*(\mu)$ if and only if, for all t_i and t_i' in T,

$$u_{t_i}[\phi^{\infty}(\rho^*(t_i'), \rho^*(\mu))] \leq u_{t_i}[\phi^{\infty}(\rho^*(t_i), \rho^*(\mu))].$$

In that case, we say that ρ^* is a limit Bayes-Nash equilibrium of the Boston mechanism given μ. Given a prior μ and strategy profile ρ, denote by $\rho(\mu)$ the induced distribution over actions.

We establish the Proposition in a series of claims.

Claim A.3. The correspondence Σ^* is non-empty and upper hemi-continuous.

Proof. Payoffs

$$u_{t_i}[\phi^{\infty}(\rho(t_i), \rho(\mu))]$$

vary continuously with σ and μ. Therefore, Σ^* is non-empty and upper hemi-continuous (see Fudenberg and Tirole (1991) p. 30). □

Claim A.4. For a fixed $\mu \in \Delta T$, the probabilities of acceptance to each school are the same in any limit Bayes Nash equilibrium.

Proof. Consider an equilibrium ρ. Let the mass of students pointing to school s in this equilibrium be

$$m_s = \sum_{t_i} \rho(t_i)(s) \cdot \mu(t_i)$$

and let the probability of acceptance at school s be p_s. Let the vectors $p = (p_s)_{s \in S}$ and $m = (m_s)_{s \in S}$. To establish the result, consider another equilibrium ρ', with associated vectors of the mass of students pointing to each school m' and probabilities of acceptance p'. Define the set of schools for which $p_s > p_s'$ as S^+ and the set of schools for which $p_s < p_s'$ as S^-.

Consider now the types who, in the equilibrium ρ, choose a school in S^+ with positive probability. All agents with types in

$$T^+ = \{t_i \in T : \max_{s \in S^+} u_{t_i} \cdot p_s > \max_{s \notin S^+} u_{t_i} \cdot p_s\}$$

must choose a school in S^+. That is, all agents who strictly prefer some school in S^+ to any school not in S^+ must point to one of the S^+schools in equilibrium. Therefore,

$$\sum_{t_i \in T^+} \mu_{t_i} \leq \sum_{s \in S^+} m_s.$$

Consider the types who choose a school in S^+ in the equilibrium ρ'. Note that the probability of obtaining entry to any school in S^+ is strictly lower at ρ' than at ρ from how we constructed S^+. Similarly, the probability of obtaining entry to any school not in S^+ is weakly higher. Therefore, in the equilibrium ρ', only agents in T^+ possibly choose a school in S^+ with positive probability. That is,

$$\sum_{s \in S^+} m'_s \leq \sum_{t_i \in T^+} \mu_{t_i}.$$

These two inequalities then imply that

$$\sum_{s \in S^+} m'_s \leq \sum_{s \in S^+} m_s.$$

However, for any $s \in S^+$ we have

$$m_s < m'_s,$$

because $p_s > p'_s$, and because probabilities of acceptance are determined by the mass of students pointing to each school. Taken together, these equations imply that $S^+ = \emptyset$. Analogously, we can prove that $S^- = \emptyset$, so $p = p'$ as desired. □

Claim A.5. P^* is non-empty, single-valued, and continuous.

Proof. The previous claims show that P^* is non-empty and single-valued. Moreover, P^* is upper hemi-continuous, because Σ^* is upper hemi-continuous and probabilities of acceptance depend continuously on equilibrium strategies and the distribution of types. Finally, P^* is continuous because continuity is equivalent to upper hemi-continuity for single-valued and non-empty correspondences. □

Claim A.6. Σ^* is convex-valued.

Proof. Fix μ, and consider two equilibria ρ and ρ', and let $\bar{\rho}$ be a convex combination of ρ and ρ'. We must show that the strategy profile $\bar{\rho}$ is an equilibrium. By Claim A.4, the probability of acceptance to each school is the same under ρ and ρ'. Therefore, the probability of acceptance is the same under $\bar{\rho}$. Because the support of $\bar{\rho}$ is contained in the union of the supports of ρ and ρ', all types play optimally under $\bar{\rho}$. □

Claim A.7. Consider a prior $\mu_0 \in \bar{\Delta} T$, and associated equilibrium ρ_0 such that, for some t_i and s_0, we have $\rho_0(t_i)(s_0) > 0$. Then there exists a neighborhood of μ_0 such that, for all μ in this neighborhood, school s_0 is optimal for t_i given $P^*(\mu)$. That is, for any $s \in S$,

$$P^*_{s_0}(\mu) \cdot u_{t_i}(s_0) \geq P^*_s(\mu) \cdot u_{t_i}(s).$$

Proof. To reach a contradiction, assume that this is not the case for some type t_i' and school s_0. Then there exists a school s_1 and sequence of priors $(\mu_k)_{k\in\mathbb{N}}$ converging to μ_0 such that, for all k,

$$P^*_{s_0}(\mu_k) \cdot u_{t_i'}(s_0) < P^*_{s_1}(\mu_k) \cdot u_{t_i'}(s_1). \tag{A.27}$$

Denote the mass of t_i' types originally pointing to school s_0 as the strictly positive constant

$$C = \rho_0(t_i')(s_0) \cdot \mu_0(t_i').$$

Denote the relative increase in probability of acceptance at school s from prior μ_0 to prior μ_k by $r_s(\mu_k) = P^*_s(\mu_k)/P^*_s(\mu_0)$. We can assume, passing to a subsequence if necessary, that the ordering of schools according to $r_s(\mu_k)$ is the same for all k. Denote the schools where the probability of acceptance increases relatively more than at school s_0 as

$$S^+ = \{s : r_s(\mu_k) > r_{s_0}(\mu_k)\}.$$

Let ρ_k be an equilibrium associated with μ_k. The mass of students pointing to schools in S^+ under ρ_k minus the mass of students pointing to schools in S^+ under ρ_0 equals

$$\sum_{s\in S^+, t_i \in T} \rho_k(t_i)(s) \cdot \mu_k(t_i) - \sum_{s\in S^+, t_i \in T} \rho_0(t_i)(s) \cdot \mu_0(t_i).$$

This sum can be decomposed as

$$\begin{aligned} &\sum_{s\in S^+, t_i \in T} (\rho_k(t_i)(s) - \rho_0(t_i)(s)) \cdot \mu_0(t_i) \\ &+ \sum_{s\in S^+, t_i \in T} \rho_k(t_i)(s) \cdot (\mu_k(t_i) - \mu_0(t_i)). \end{aligned} \tag{A.28}$$

Students who point to schools in S^+ under ρ_0 continue to do so under ρ_k. And, because equation (A.27) holds, the mass of students who point to schools in $S \backslash S^+$ under ρ_0 but who point to schools in S^+ under ρ_k is at least C. Hence, the first term in expression (A.28) is bounded below by C. Moreover, the second term converges to 0, because μ_k converges to μ_0. Therefore, for large enough k, the mass of students pointing to schools in S^+ under ρ_k is strictly larger than the mass of students pointing to schools in S^+ under ρ_0.

This implies that there exists a school $s^+ \in S^+$ such that the mass of students pointing to s^+ is strictly greater under ρ_k than under ρ_0. And there exists a school $s^- \in S \backslash S^+$ such that the mass of students pointing to s^- is strictly smaller under ρ_k than under ρ_0. However, from the way we constructed S^+ we have that $r_{s^+}(\mu_k) > r_{s^-}(\mu_k)$, which is a contradiction. □

Claim A.8. Consider a prior μ_0, and associated equilibrium ρ_0 such that, for some t_i and school s_0, the mass of students pointing to s_0 is strictly lower than its capacity:

$$\sum_{t_i \in T} \rho_0(t_i)(s_0) \cdot \mu_0(t_i) < q_{s_0}.$$

Then there exists a neighborhood of μ_0 such that, for all μ in this neighborhood, $P^*_{s_0}(\mu) = 1$.

Proof. Denote the excess supply of school s_0 as the strictly positive constant

$$C = q_{s_0} - \sum_{t_i \in T} \rho_0(t_i)(s_0) \cdot \mu_0(t_i).$$

To reach a contradiction, assume that the claim's conclusion does not hold. Then there exists a sequence of priors $(\mu_k)_{k \in \mathbb{N}}$ converging to μ_0 such that, for all k, $P^*_{s_0}(\mu_k) < 1$. Let ρ_k be an equilibrium given μ_k. The fact that the probability of acceptance at s_0 is lower than 1 under ρ_k implies that the difference between the mass of students pointing to s_0 under ρ_k and ρ_0 is bounded below by C. That is,

$$\sum_{t_i \in T} \rho_k(t_i)(s_0) \cdot \mu_k(t_i) - \sum_{t_i \in T} \rho_0(t_i)(s_0) \cdot \mu_0(t_i) > C.$$

Because μ_k converges to μ_0, this implies that, for large enough k,

$$\sum_{t_i \in T} (\rho_k(t_i)(s_0) - \rho_0(t_i)(s_0)) \cdot \mu_0(t_i) > C/2. \tag{A.29}$$

As in the previous claim's proof, denote the relative increase in the probability of acceptance at school s from prior μ_0 to prior μ_k by $r_s(\mu_k) = P^*_s(\mu_k)/P^*_s(\mu_0)$. We can assume, passing to a subsequence if necessary, that the ordering of schools according to $r_s(\mu_k)$ is the same for all k. Denote the set of schools where the relative probability of acceptance does not increase more than in s_0 by

$$S^- = \{s : r_s(\mu_k) \leq r_{s_0}(\mu_0)\} \backslash \{s_0\}.$$

All students who point to a school in $S^- \cup \{s_0\}$ under ρ_k point to schools in $S^- \cup \{s_0\}$ under ρ_0. Thus,

$$\sum_{s \in S^- \cup \{s_0\}, t_i \in T} (\rho_k(t_i)(s) - \rho_0(t_i)(s)) \cdot \mu_0(t_i) \leq 0.$$

Substituting inequality (A.29) we have that, for large enough k,

$$\sum_{s \in S^-, t_i \in T} (\rho_k(t_i)(s) - \rho_0(t_i)(s)) \cdot \mu_0(t_i) < -C/2. \tag{A.30}$$

The mass of students pointing to schools in S^- under ρ_k minus the mass of students pointing to schools in S^- under ρ_0 equals

$$\sum_{s \in S^-, t_i \in T} \rho_k(t_i)(s) \cdot \mu_k(t_i) - \sum_{s \in S^-, t_i \in T} \rho_0(t_i)(s) \cdot \mu_0(t_i).$$

This sum can be decomposed into

$$\begin{aligned} &\sum_{s \in S^-, t_i \in T} (\rho_k(t_i)(s) - \rho_0(t_i)(s)) \cdot \mu_0(t_i) \\ &+ \sum_{s \in S^-, t_i \in T} \rho_k(t_i)(s) \cdot (\mu_k(t_i) - \mu_0(t_i)). \end{aligned}$$

By inequality (A.30), for large enough k, the first term in the expression above is smaller than $-C/2$. Because the second term converges to 0, we have that, for sufficiently large k, the mass of students pointing to schools in S^- under ρ_k is strictly lower than the mass of students pointing to schools in S^- under ρ_0. Hence, for at least one school s^- in S^-, we have $r_{s^-}(\mu_k) \geq 1$. But this contradicts $r_{s^-}(\mu_k) \leq r_{s_0}(\mu_k) < 1$. □

Claim A.9. The correspondence Σ^* is lower hemi-continuous in $\bar{\Delta} T$.

Proof. To prove lower hemi-continuity, fix μ_0, an associated limit equilibrium ρ_0, and consider a sequence $(\mu_k)_{k \geq 1}$ converging to μ_0. Fix $\epsilon > 0$. We will show that there exists a sequence of equilibria $(\rho_k)_{k \geq 1}$, associated with the μ_k, which converges to a strategy profile with distance lower than ϵ to ρ_0.

Part 1: Define the candidate sequence of equilibria.

Let ρ'_k be an equilibrium associated with μ_k. Passing to a subsequence, we can assume that $(\rho'_k)_{k\geq 1}$ converges to an equilibrium ρ'_0 associated with μ_0. Define

$$\rho_k(t_i) = \rho'_k(t_i) + (1-\epsilon)\cdot[\rho_0(t_i) - \rho'_0(t_i)]\cdot \frac{\mu_0(t_i)}{\mu_k(t_i)}.$$

Note that this sequence converges to $\epsilon\cdot\rho'_0 + (1-\epsilon)\cdot\rho_0$. Hence, it converges to a point within ϵ distance from ρ_0.

Part 2: For large enough k, ρ_k is a strategy profile.

Because the sum $\sum_s \rho_k(t_i)(s) = 1$, we only have to demonstrate that every $\rho_k(t_i)(s)$ is nonnegative. To see this, note that ρ_k converges to $\epsilon\cdot\rho'_0 + (1-\epsilon)\cdot\rho_0$. Hence, if either $\rho_0(t_i)(s) > 0$ or $\rho'_0(t_i)(s) > 0$, then $\rho_k(t_i)(s) > 0$ for sufficiently large k. The remaining case is when $\rho_0(t_i)(s) = \rho'_0(t_i)(s) = 0$. In this case we have that $\rho_k(t_i)(s) = \rho'_k(t_i)(s) \geq 0$.

Part 3: For sufficiently large k, the ρ_k are equilibria.

We will begin by proving that, for sufficiently large k, the probabilities of acceptance under ρ_k equal those under ρ'_k. That is, the probabilities of acceptance under ρ_k equal $P^*(\mu_k)$. To see this, note that the mass of agents pointing to school s under ρ_k equals

$$\sum_{t_i}\rho_k(t_i)(s)\cdot\mu_k(t_i) = \sum_{t_i}\rho'_k(t_i)(s)\cdot\mu_k(t_i) + (1-\epsilon)\cdot\sum_{t_i}[\rho_0(t_i)(s) - \rho'_0(t_i)(s)]\cdot\mu_0(t_i). \quad \text{(A.31)}$$

There are two cases. The first case is when the mass of students pointing to s is strictly lower than q_s under either ρ_0 or ρ'_0. In this case, we have $P^*_s(\mu_0) = 1$, so that, in the mass of students pointing to s is at most equal to q_s under both ρ'_0 and ρ_0. The mass of students pointing to school s under ρ_k converges to

$$\epsilon\cdot(\sum_{t_i\in T}\rho'_0(t_i)(s)) + (1-\epsilon)\cdot(\sum_{t_i\in T}\rho_0(t_i)(s)).$$

That is, to an average of the mass of students pointing to s under ρ'_0 and ρ_0. Because both quantities are weakly smaller than q_s, and at least one of them is strictly lower than q_s, this average is strictly lower than q_s. Thus, for large enough k, the probability of acceptance to s under ρ_k is 1. This is equal to the probability of acceptance under ρ'_k, by Claim A.8.

The second case is when the mass of students pointing to school s is at least equal to q_s both under ρ_0 and under ρ'_0. If this is the case, then the mass of students pointing to school s is the same under ρ_0 and under ρ'_0, because probabilities of acceptance are the same in any equilibrium under μ_0. Therefore, the sum

$$\sum_{t_i}[\rho_0(t_i)(s) - \rho'_0(t_i)(s)] \cdot \mu_0(t_i) = 0.$$

Substituting this in Equation (A.31), we have that the probabilities of acceptance under ρ_k and ρ'_k are equal, as desired.

To complete the proof we show that, for large enough k, the strategies ρ_k are optimal given $P^*(\mu_k)$. Consider a school s with $\rho_k(t_i)(s) > 0$. Therefore, either $\rho'_k(t_i)(s) > 0$ or $\rho_0(t_i)(s) > 0$. If $\rho'_k(t_i)(s) > 0$, then it is optimal for type t_i to point to s under $P^*(\mu_k)$, because ρ'_k is an equilibrium. Likewise, if $\rho_0(t_i)(s) > 0$, then Claim A.7 implies that, for large enough k, it is optimal for type t_i to report s under $P^*(\mu_k)$. □

The proposition then follows from Claims A.3, A.5, and A.9.

References

Fudenberg, Drew and Jean Tirole, *Game Theory*, Cambridge, MA: MIT Press, 1991.